Statistical Physics & Condensed Matter Theory I: Exercise

A (classical) field theory for the Ising model

Consider a square lattice in \( d \) dimensions. On each lattice site lives an Ising spin \( S_a \), which is a classical spin variable taking only two possible values, \( S_a = \pm 1 \). The vector \( a \) labels the lattice sites. The Ising model is defined by the classical Hamiltonian

\[
H_I = -\sum_{a,b} J_{ab} S_a S_b - \sum_a H_a S_a,
\]

where \( H_a \) is a magnetic field (which can vary from one position to another, hence the index), and the interaction coefficients \( J_{ab} \equiv J(|a - b|) \) fall off quickly with distance.

The classical partition function of the Ising model can be written as the sum over all possible Ising spin configurations,

\[
Z_I = \sum_{\{S_a\}} e^{-\beta H_I} = \sum_{\{S_a\}} e^{\sum_{a,b} K_{ab} S_a S_b + \sum_a h_a S_a}.
\]

where we have defined \( K_{ab} \equiv \beta J_{ab} \) and \( h_a \equiv \beta H_a \).

a) Let’s consider the noninteracting problem first: we set \( K_{ab} = 0 \) everywhere. Show that the classical partition function then becomes

\[
Z_I|_{K_{ab}=0} = \prod_a 2 \cosh h_a.
\]

b) What prevents us from solving this exactly\(^1\) is the presence of the interaction term. We therefore will ‘decouple’ this interaction term using a Hubbard-Stratonovich transformation. Explicitly: starting from a representation of the unit operator as a field integral over an auxiliary (real, bosonic) field \( \psi \),

\[
1 \equiv \mathcal{N} \int \mathcal{D}\psi e^{-\frac{1}{4} \sum_{ab} \psi_a (K^{-1})_{ab} \psi_b}, \quad \mathcal{D}\psi \equiv \prod_a d\psi_a
\]

with \( K^{-1} \) the matrix inverse of \( K \) (and the normalization \( \mathcal{N} = \left[ \det (4\pi K^{-1}) \right]^{1/2} \)), show that the partition function of the interacting Ising theory can be written

\[
Z_I = \mathcal{N} \int \mathcal{D}\psi \sum_{\{S_a\}} e^{-\frac{1}{4} \sum_{ab} \psi_a (K^{-1})_{ab} \psi_b + \sum_a (\psi_a + h_a) S_a}
\]

\(^1\)Note that the solution to the \( d = 1 \) Ising theory (in any field \( h_a = h \)) is straightforward; in \( d = 2 \), Onsager offered the solution at zero field; the \( d = 3 \) case remains one of the most famous outstanding problems of classical statistical physics.
e) Perform the summation over all possible spin configurations \( \sum (S_a) \), and show that (after a redefinition of the field \( \phi_a \equiv \frac{1}{2} \sum_b (K^{-1})_{ab} (\psi_b + h_b) \), inessential constant terms being reabsorbed into a redefinition of the constant \( \mathcal{N} \to \tilde{\mathcal{N}} \)) we get to the representation

\[
Z_I = \tilde{\mathcal{N}} \int \mathcal{D}\phi e^{-S[\phi]}, \quad S[\phi] \equiv \sum_{ab} \phi_a K_{ab} \phi_b - \sum_a \phi_a h_a - \sum_a \ln \cosh(2\sum_b K_{ab} \phi_b)
\]

\[
d) \text{ We now make a low-temperature approximation, by assuming that the fluctuations of } \phi \text{ are small (for our purposes here, this translates into taking } |\phi_a| \ll 1). \text{ We define the Fourier representations (} N \text{ is the total number of Ising spins in the system, and we assume periodic boundary conditions in all directions for simplicity) of the field and interaction as}
\]

\[
\phi_a = \frac{1}{\sqrt{N}} \sum_k e^{-ik \cdot a} \phi_k, \quad K_{ab} = \frac{1}{N} \sum_k e^{-i(k-a) \cdot b} K_k,
\]

and assume that the interaction is mostly local in space, so that we can use a small-momentum expansion (also called the gradient expansion) of the interaction, \( K_k = K_0 + \frac{1}{2} k \cdot k K_0'' + O(|k|^4) \). Using this and the expansion \( \ln \cosh x = \frac{1}{2} x^2 - \frac{1}{12} x^4 + \ldots \), show that the action has the form

\[
S[\phi] = \sum_k [\phi_k (c_1 + c_2 k \cdot k) \phi_{-k} + c_3 \phi_k h_{-k}] + \frac{c_4}{N} \sum_{k_1, k_2, k_3, k_4} \phi_{k_1} \phi_{k_2} \phi_{k_3} \phi_{k_4} \delta_{k_1+k_2+k_3+k_4,0} + \ldots
\]

(obtain an expression for the coefficients \( c_i \)).

e) Upon Fourier transforming back to real space and a rescaling (you can put the field \( h \) to zero), the action becomes the so-called \( \phi^4 \) theory,

\[
Z = \int \mathcal{D}\phi e^{-S[\phi]}, \quad S[\phi] = \int d^d x \left[ \frac{1}{2} (\partial \phi)^2 + \frac{r}{2} \phi^2 + g \phi^4 \right].
\]

Give the coefficients \( r \) and \( g \) in terms of the coefficients \( J_0 \equiv \beta^{-1} K_0 \) and \( J_0'' \equiv \beta^{-1} K_0'' \). At what temperature does the Ising model have a transition? For your information: the Ising model has a transition when \( r \) changes sign from positive values to negative ones, i.e. when the potential becomes a ‘Mexican hat’ like potential. This is the field-theoretic way of describing this transition.