A (classical) field theory for the Ising model

a) Trivial since $\sum S^a e^{-\beta h^a S^a} = 2 \cosh h^a$.

b) We here consider the given representation of unity, but for a shifted value of the field:

$$1 \equiv N \int \mathcal{D}\psi e^{-\frac{1}{4} \sum_{ab} \varphi_{ab} (K^{-1})_{ab} \varphi_{ab} + \frac{1}{2} \sum_{ab} \theta_a (K^{-1})_{ab} \theta_b}$$

in which $\theta_a$ are constants to be chosen (the integration measure doesn’t change under such shifts by constants). Written out explicitly,

$$1 \equiv N \int \mathcal{D}\psi e^{-\frac{1}{4} \sum_{ab} \varphi_{ab} (K^{-1})_{ab} \varphi_{ab} + \frac{1}{2} \sum_{ab} \theta_a (K^{-1})_{ab} \theta_b}$$

Choosing $\theta_a = -2 \sum_b K_{ab} S^b$ gives

$$e^{\sum_{ab} S^a K_{ab} S^b} = N \int \mathcal{D}\psi e^{-\frac{1}{4} \sum_{ab} \varphi_{ab} (K^{-1})_{ab} \varphi_{ab} + \sum_a S^a \varphi_a}$$

which gives the required answer by direct substitution.

c) Simple.

d) Consider

$$x_a = 2 \sum_b K_{ab} \phi_b = \frac{2}{N^{3/2}} \sum_{k,k'} \sum_b e^{-i(k-a-b) - i(k'-b)} K_k \phi_k = \frac{2}{\sqrt{N}} \sum_k e^{-i k^a} K_k \phi_k$$ (1)

We can then write

$$\sum_a \frac{x^2_a}{2} = \frac{2}{N} \sum_{k_1,k_2} K_{k_1} K_{k_2} \phi_{k_1} \phi_{k_2} \sum_a e^{-i(k_1 + k_2)^a} \phi_{k'} = 2 \sum_k K_{-k} K_{k} \phi_{-k}$$ (2)

and

$$\sum_a \frac{x^4_a}{12} = \frac{4}{3N^2} \sum_{k_1,...,k_4} K_{k_1} K_{k_2} K_{k_3} K_{k_4} \phi_{k_1} \phi_{k_2} \phi_{k_3} \phi_{k_4} \sum_a e^{-i(k_1 + k_2 + k_3 + k_4)^a}$$ (3)

Doing the gradient expansion in each of these, and keeping only leading terms,

$$\sum_a \frac{x^2_a}{2} \approx 2 \sum_k (K_0 + \frac{1}{2} |k|^2 K_0'' + ...) \phi_k \phi_{-k}$$ (4)
Putting things together, we get the form for the action given in the question, with coefficients
\[ c_1 = K_0 - 2K_0^2, \quad c_2 = \frac{1}{2}K_0'' - 2K_0K_0', \quad c_3 = -1, \quad c_4 = \frac{4}{3}K_0^4 \]  

(6)

e) Back to real space (specializing to \( h = 0 \))

\[ S[\phi] = \int d^d x \left[ c_2 (\partial \phi)^2 + c_1 \phi^2 + c_4 \phi^4 \right] \]  

(7)

Rescaling \( \phi \to \frac{1}{\sqrt{2c_2}} \phi \), we get \( r = \frac{c_1}{c_2} \) and \( g = \frac{c_4}{4c_2^2} \).

The calculation only makes sense if \( c_2 > 0 \) so \( K_0 < 1/4 \). \( r \) changes sign when \( c_1 = 0 \) so \( K_0 = 1/2 \) and thus the critical temperature is given by the condition \( \beta_c = \frac{1}{2J_0} \). For \( \beta > \beta_c \), \( r < 0 \) and the system is in the ordered phase. For \( \beta < \beta_c \), \( r > 0 \) and the system is in the disordered phase.