Harmonic oscillator and coherent states

Consider a single harmonic oscillator, in units where $\hbar = 1$, $m = 1$ and $\omega = 1$. The Hamiltonian is thus simply

$$H = a^\dagger a + 1/2.$$ 

As usual, the bosonic annihilation and creation operators, defined as $a = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p})$, $a^\dagger = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p})$, satisfy the canonical commutation relation $[a, a^\dagger] = 1$. We denote the ground state as $|0\rangle$.

a) Is the ground state $|0\rangle$ a coherent state? Using the $a, a^\dagger$ operators, compute the averages $\bar{x} = \langle 0 | \hat{x} | 0 \rangle$ and $\bar{p} = \langle 0 | \hat{p} | 0 \rangle$. Show that the variances $\Delta x = \sqrt{\langle \hat{x}^2 \rangle - \bar{x}^2}$, $\Delta p = \sqrt{\langle \hat{p}^2 \rangle - \bar{p}^2}$ saturate (in other words, optimize) Heisenberg’s uncertainty relation $\Delta x \Delta p \geq \frac{1}{2}$.

b) The translation operator $T(\alpha) = e^{-\alpha \frac{\hat{p}}{\hbar}} = e^{-i\alpha \hat{p}}$ shifts the average position of a wavefunction¹. Show that the shifted ground state $T(\alpha)|0\rangle$ behaves like a coherent state (explicitly: show that it is an eigenstate of the annihilation operator, and give the eigenvalue) ². What is this state’s normalization? Compute again the averages $\bar{x}, \bar{p}$ and variances. Is Heisenberg’s relation still optimized?

c) Consider now the more general state $|z\rangle = e^{za^\dagger - z^* a} |0\rangle$, in which $z$ is an arbitrary complex number. Is this state normalized? Does it behave like a coherent state? Repeat the calculation of $\bar{x}, \bar{p}$ and variances. Is Heisenberg’s relation still optimized? ³

¹This is simply because, in first quantization, we have (using a Taylor series expansion) $\psi(x - \alpha) = \psi(x) - \alpha \frac{\partial}{\partial x} \psi(x) + \frac{\alpha^2}{2!} \frac{\partial^2}{\partial x^2} \psi(x) + ... = e^{-\alpha \frac{\partial}{\partial x}} \psi(x)$.

²Hint: rewrite everything in terms of $a, a^\dagger$. You will need identities coming from the Campbell-Baker-Hausdorff formula; these are given in the ‘Useful formulas’ at the end.

³Extra info, *not* important for the question: an important set of states, useful in the fields of quantum optics, gravitational waves and many others, are the squeezed states. These are given by operating on the vacuum with the squeeze operator $S(z) \equiv e^{\frac{1}{2} z(a^\dagger)^2 - \frac{1}{2} z^* a^2}$. Unlike coherent states, which have $\Delta x = \Delta p$, squeezed states have $\Delta x \neq \Delta p$ (thereby the name), and do not saturate Heisenberg’s relation at all times, but only at some specific ones.
Useful formulas

Campbell-Baker-Hausdorff formula

Let $A$ and $B$ be two quantum operators such that $[A, B]$ commutes with $A$ and $B$. Then, the following identities hold:

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}, \quad [A, e^{\lambda B}] = \lambda [A, B] e^{\lambda B}.$$ 

These are a consequence of a more general identity called the Campbell-Baker-Hausdorff formula.