Statistical Physics & Condensed Matter Theory I:  
Exercise

Grassmann variables

We have seen in class the (multivariable) Gaussian integration identities for bosonic variables (real case on the left, complex case on the right):

\[ \int d\mathbf{v} e^{-\frac{1}{2} \mathbf{v}^T \mathbf{A} \mathbf{v}} = (2\pi)^{N/2} \det \mathbf{A}^{-1/2}, \]

\[ \int d\mathbf{w}^\dagger \mathbf{w} e^{-\mathbf{w}^\dagger \mathbf{B} \mathbf{w}} = \pi^N \det \mathbf{B}^{-1/2}, \]

where (for the left part) \( \mathbf{A} \) is a positive definite real symmetric \( N \)-dimensional matrix, \( \mathbf{v} \) is an \( N \)-component real vector, \( d\mathbf{v} \equiv \prod_i dv_i \), and (for the right part) \( \mathbf{w} \) an \( N \)-dimensional complex vector, \( d(\mathbf{w}^\dagger, \mathbf{w}) \equiv \prod_{i=1}^N d\Re w_i d\Im w_i \), and \( \mathbf{B} \) a complex matrix with positive definite Hermitian part.

For the case of Grassmann variables, we have only the equivalent of the ‘right’ part, namely

\[ \int d(\bar{\phi}, \phi) e^{-\bar{\phi}^T \mathbf{B} \phi} = \det \mathbf{B} \]

where \( \bar{\phi} \) and \( \phi \) are independent \( N \)-component vectors of Grassmann variables, the measure is \( d(\bar{\phi}, \phi) \equiv \prod_{i=1}^N d\bar{\phi}_i d\phi_i \), and \( \mathbf{B} \) can be an arbitrary complex matrix.

In this exercise, we complete the table by providing the missing ??? piece in the above equation. Consider thus the following multidimensional Grassmann integration

\[ I_G \equiv \int d\phi e^{-\frac{1}{2} \sum_{i,j=1}^N \phi_i A_{ij} \phi_j} \]

(take for definiteness \( d\phi \equiv \int d\phi_N \int d\phi_{N-1} \ldots \int d\phi_1 \)), where \( \mathbf{A} \) is taken to be an antisymmetric matrix (no points: do you see why ?).

a) Compute this explicitly for \( N = 1, 2 \) and 3.

b) Show that this vanishes for any matrix if \( N \) is odd.

c) Show that for \( N \) even, \( I_G = (-1)^{N/2} \text{Pf} \mathbf{A} \) where

\[ \text{Pf} \mathbf{A} \equiv \frac{1}{2^{N/2}(N/2)!} \sum_{i_1, i_2, \ldots, i_N=1}^N \epsilon_{i_1 i_2 i_3 \ldots i_N} A_{i_1 i_2} A_{i_3 i_4} \ldots A_{i_{N-1} i_N} \]

(in which \( \epsilon_{i_1 i_2 \ldots} \) is the completely antisymmetric tensor in all indices, with \( \epsilon_{12 \ldots N} \equiv 1 \)) is called the Pfaffian of matrix \( \mathbf{A} \).

For your information: the Pfaffian of an antisymmetric matrix is such that \( (\text{Pf} \mathbf{A})^2 = \det \mathbf{A} \) (an identity you do not have to show here !), so this does offer a parallel to the ‘square root of determinant’ of the bosonic case.