$\mathcal{N} = 4$ Super Yang-Mills and spin chains

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Abstract

In this paper we begin by providing a brief overview of Poincare symmetry,conformal symmetry and supersymmetry, discussing their algebras and their representations. We then introduce $\mathcal{N} = 4$ Super Yang-Mills theory, discuss the gauge invariant operators of this theory and explain what is the planar limit. We compute the one-loop anomalous dimension for single trace scalar operators and show that the result we get can be mapped to an $SO(6)$ spin chain. We do the mapping for the $SU(2)$ sector, where the spin chain is the Heisenberg one, the eigenvalues of which can be computed by the Bethe equations.

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1 Introduction and Motivation

One of the greatest breakthroughs occurred in theoretical physics in the last twenty years is the Anti-de Sitter/Conformal Field Theory correspondence, which is a duality relating a conformal field theory to a string theory. The first and most important example of an AdS/CFT correspondence was that $\mathcal{N}$=4 Super Yang Mills (SYM) theory is dual to the type IIB superstring on the $AdS_5 \times S^5$ background. One important implication of this is that a string theory can be mapped to a gauge theory and vice versa.

This idea actually is not so strange. Firstly, let's recall that the first attempt to understand strong interaction was via string theory. Of course, this idea didn't work and today the strong interaction is described with an SU(3) gauge theory, the Quantum Chromodynamics (QCD). However, some aspects of this, like the flux tubes of the QCD fields, appear to have an effective description as strings.

But the first realization of an actual correlation between string theories and gauge theories was made by 't Hooft. He proposed that a gauge theory (with gauge group SO(N), SU(N)), in the planar limit, in the large N perturbative expansion, can be understood as a genus expansion similar to the one that comes from the Feynman diagrams of string theory. According to this, we can define $\lambda \equiv g_{YM}^2 N$ (g_{YM} is the coupling constant of the gauge theory) that indicates the number of quantum loops.

In the context of the AdS/CFT correspondence, we have:

$$\frac{1}{N} = \frac{g_s}{4\pi T^2}, \quad \lambda = 4\pi^2 T^2 \tag{1.1}$$

Note that g_s is the coupling constant of string theory and T is the (effective) tension of the string.

Let's see more about these parameters. In particular, if we consider $\lambda \ll 1$, we will have the weak coupling regime for the gauge theory. We can acquire reliable results based on conventional treatment of some QFT, e.g. first loop Feynman diagrams calculations etc. On the other hand, for $\lambda \to \infty, g_s \ll 1$, we have the strong coupling regime. This is the region, where the perturbative string theory is applicable. Therefore, we can rely only on results that are near to this limit.

Exactly at this point, we find the first difficulty of testing the conjecture. It maps essentially two regions that cannot be tested simultaneously. When we can have credible data for the one part of the duality, we fail completely in getting insight for the other one. This is what is called weak/strong coupling duality. The whole situation and the incompatibility of the two sides are depicted nicely in Figure 1.

A solution to these problem comes from integrability notion. Just by taking the planar limit ($N \to \infty$) in $\mathcal{N}$=4 SYM, someone can get information for arbitrary λ. So we can relate and do calculations for different quantities in both theories.

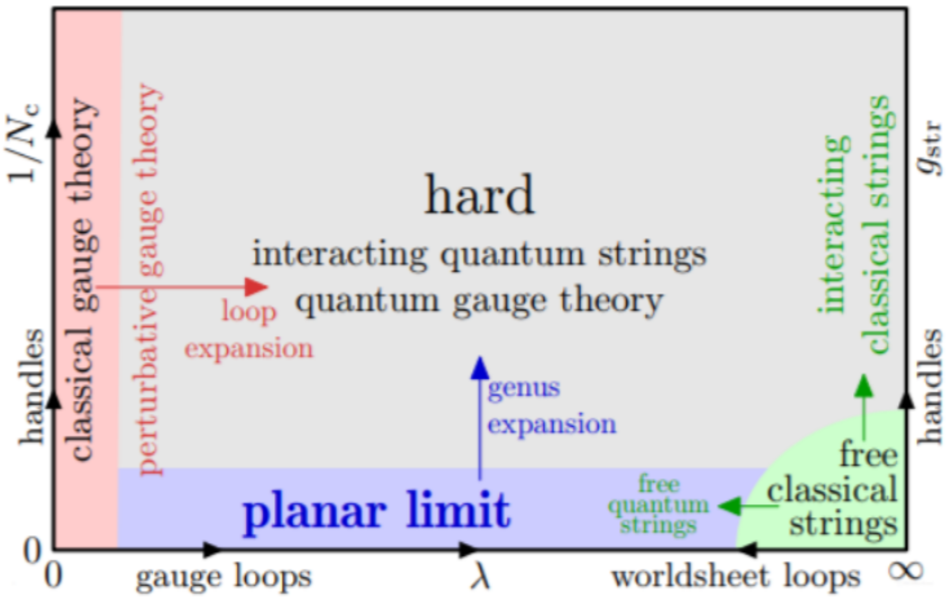

Figure 1: Parameter phase space and the effective region of each description

2 $\mathcal{N} = 4$ Super Yang-Mills Theory

In order to facilitate the understanding of the symmetry group of $\mathcal{N} = 4$ SYM , its implications to what we are doing in the present paper, as well as the field content of $\mathcal{N} = 4$ Super Yang Mills, we will present here some necessary background material concerning group theory, representation theory and supersymmetry.

The full group of $\mathcal{N} = 4$ SYM is denoted by $PSU(2, 2|4)$ and includes Poincare symmetry (Lorentz transformations and translations), conformal symmetry, supersymmetry and R-symmetry. We begin by discussing the Poincare group.

2.1 Poincare symmetry

The Poincare group is a Lie group consisting of translations and the Lorentz group (rotations and boosts that leave the Minkowski metric invariant). Its Lie algebra has 10 generators, 6 generators for the $SO(1, 3)$ Lorentz group $J_{\mu\nu}$ ($J_{\mu\nu}$ is antisymmetric in its indices, hence the 6 independent components) and 4 for the translations P_μ. They satisfy the following commutation relations:

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\mu\nu}J_{\nu\sigma} + \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\nu\rho}J_{\mu\sigma} - \eta_{\mu\sigma}J_{\nu\rho}) \tag{2.1}$$

$$[J_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu), \qquad [P_\mu, P_\nu = 0] \tag{2.2}$$

Since particles correspond to unitary irreducible representations of the Poincare group, we want to classify the unitary representations of the Poincare group. The Poincare group is a non-compact group, since the boosts and the translations are specified by parameters that do not take values on a compact interval (closed and bounded). For a non-compact group, apart from the trivial representation there is no other unitary finite-dimensional representation and so the representations have to be labelled by continuous parameters, which in our case can be the momentum p^μ. Then, we have to fix the non-compact transformations by choosing a specific frame. The group that leaves a specific p^μ invariant is called the little group. For massive particles we can choose the frame $p^\mu = (m, 0, 0, 0)$. In order to classify the irreducible representations, it is useful to find the Casimir operators, which are members of the Lie algebra which commute with all other members. For the Poincare group $P_\mu P^\mu$ and $W_\mu W^\mu$ are the Casimir operators, where $W_\mu = \frac{1}{2}\epsilon_{\kappa\nu\rho\mu}J^{\kappa\nu}P^\rho$ is the Pauli-Lubanski vector. Massive particles can be classified by their spin $W^2 = j(j+1)$ and since one component of W_μ commutes with P_μ, by one of the spin components. So, we can denote the massive particle states as $|p^\mu, j, j_3>$, where for a given j there are $2j+1$ states. For massless particles we choose $p^\mu = (E, 0, 0, E)$. The little group now is generated by $M_1 = J_{10} + J_{13}$, $M_2 = J_{20} + J_{23}$ and J_{12}. The first two are non compact generators, since J_{10} and J_{20} generate boosts, which means that they will be trivially represented. So, the particle states are labelled by just one number, the helicity λ which is the eigenvalue of J_{12} and turns out to be an integer or a half-integer.

2.2 Conformal Symmetry

We next consider conformal transformations, which are transformations that leave the metric invariant up to an arbitrary positive spacetime dependent factor. In addition to the Poincare generators, we have also now the generator of dilatations D and the generators of the special conformal generators K_μ. These satisfy the following commutation relations:

$$[J_{\mu\nu}, K_\rho] = i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu), \qquad [K_\mu, K_\rho] = 0 \tag{2.3}$$

$$[D, P_\mu] = iP_\mu, \qquad [D, K_\mu] = -iK_\mu, \qquad [D, J_{\mu\nu}] = 0 \tag{2.4}$$

$$[K_\mu, P_\nu] = -2i(\eta_{\mu\nu}D - J_{\mu\nu}) \tag{2.5}$$

In a conformal field theory, local operators transform in irreducible representations of the conformal algebra. An operator with dimension Δ satisfies:

$$[D, \mathcal{O}(x)] = i\left(-\Delta - x\frac{\partial}{\partial x}\right)\mathcal{O}(x) \tag{2.6}$$

This means that under a dilatation $x \to \lambda x$, it transforms as $\mathcal{O}(x) \to \lambda^{-\Delta}\mathcal{O}(\lambda x)$. We can consider only the operators at $x = 0$ and then apply P^μ to shift the position in any point x. Using the Jacobi identity we can show that K_μ decreases the scaling dimension by one,while similarly P_μ increases it. Since we are interested in unitary representations, there has to be a lower bound on the scaling dimension. The fields that satisfy this bound are given by:

$$[K_\mu, \mathcal{O}] = 0 \tag{2.7}$$

and are called primary fields, whereas the higher dimensional fields created by acting with P_μ on the primaries are called descendants.

The conformal symmetry determines the form of the two- and three- point functions up to constants. Since in this paper we only use two-point functions, we will only refer to them. Lorentz and translation invariance requires that the correlator $\langle\phi(x_1)\phi(x_2)\rangle$ depends only on $(x_1 - x_2)^2$. Since the correlator under a scaling transformation transforms as $\langle\phi(x_1)\phi(x_2)\rangle = \lambda^{\Delta_1+\Delta_2}\langle\phi(\lambda x_1)\phi(\lambda x_2)\rangle$, scaling invariance requires moreover that the general form of the correlator is:

$$\langle\phi(x_1)\phi(x_2)\rangle = \frac{C}{(x_1 - x_2)^{\Delta_1+\Delta+2}} \tag{2.8}$$

Conformal invariance $\langle 0|[K_\mu, \phi_1\phi_2]|0\rangle = 0$ implies in addition that $\Delta_1 = \Delta_2$. Finally, it is always possible to diagonalize the constant C in the space of scalar primary operators, so that the two-point function is zero unless $\phi_2 = \bar{\phi}_1$.

2.3 Supersymmetry

Supersymmetry relates bosons to fermions, since under a supersymmetry transformation bosonic and fermionic degrees of freedom are exchanged. The supersymmetry algebra is a graded extension of the Poincare Lie algebra. In addition to the usual generators of the Poincare algebra, one also considers one or more anticommuting fermionic generators. Each generator is assigned a grade. Bosonic generators have grade 0, while fermionic ones have +1. In general a product of fields has as grade the sum of the individual grades of the fields modulo 2. The commutation relation between two generators with grades g_1 and g_2 is given by:

$$[G_1, G_2\} = G_1G_2 - (-1)^{g_1g_2}G_2G_1 \tag{2.9}$$

Let us first consider the case of one independent supersymmetry. This means that we have to add a spinor supercharge Q. We will use Weyl notation and write Q_α and $\bar{Q}_{\dot{\alpha}}$ for the left-handed and the right-handed spinors respectively. The greek indices α and $\dot{\alpha}$ take values $1, 2$. As usual, these Weyl spinors transform in the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations of the Lorentz group. The most general superalgebra for $\mathcal{N} = 1$, where $\mathcal{N}$ is the number of independent supersymmetries is:

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}}P_\mu, \qquad \{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0 \tag{2.10}$$

$$[Q_\alpha, J^{\mu\nu}] = (\sigma^{\mu\nu})_\alpha^{\ \beta}Q_\beta, \qquad [\bar{Q}_{\dot{\alpha}}, J^{\mu\nu}] = \epsilon_{\dot{\alpha}\beta}(\bar{\sigma}^{\mu\nu})^\beta_{\ \dot{\gamma}}\bar{Q}^{\dot{\gamma}} \tag{2.11}$$

$$[Q_\alpha, P^\mu] = 0, \qquad [\bar{Q}_{\dot{\alpha}}, P^\mu] = 0 \tag{2.12}$$

where $\sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)$. In the above expressions, the first line comes from the fact that the product of a chiral and an antichiral spinor gives rise to a vector. The second line shows what we have already said, namely that Q_α and $Q_{\dot{\alpha}}$ are chiral and antichiral spinors respectively. Finally, the third line can be proved using the Jacobi identity, which should be appropriately modified by replacing commutators with anticommutators where necessary.

The superalgebra is also invariant under the action of the $U(1)$ group on the charges:

$$Q_\alpha \to Q'_\alpha = e^{ia}Q_\alpha, \qquad \bar{Q}_{\dot{\alpha}} \to \bar{Q}'_{\dot{\alpha}} = e^{-ia}\bar{Q}_{\dot{\alpha}} \tag{2.13}$$

and the generator of $U(1)$ obeys the following commutation relations:

$$[Q_\alpha, R] = Q_\alpha, \qquad [\bar{Q}_{\dot{\alpha}}, R] = -\bar{Q}_{\dot{\alpha}} \tag{2.14}$$

If we now consider more than one independent supersymmetries, we will have Q^a_α and $\bar{Q}^a_{\dot\alpha}$ spinors and antispinors, where $a = 1, 2, ..., \mathcal{N}$. The commutation relations of the last two lines after adding the a index will be the same, while the first line will become:

$$\{Q^a_\alpha, \bar{Q}_{b\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^a_b, \quad \{Q^a_\alpha, Q^b_\beta\} = 0, \quad \{\bar{Q}_{a\dot\alpha}, \bar{Q}_{b\dot\beta}\} = 0 \tag{2.15}$$

The superalgebra possesses now the group of automorphisms $U(N)$ (in four spacetime dimensions), which acts on the charges as:

$$Q^a_\alpha \to Q^{a'}_\alpha = R^a{}_b Q^b_\alpha, \qquad \bar{Q}_{a\dot\alpha} \to \bar{Q}'_{a\dot\alpha} = \bar{Q}_{b\dot\alpha}(R^\dagger)^b{}_a \tag{2.16}$$

where R^b_a are $N \times N$ matrices. This global symmetry of the supersymmetry algebra is called R-symmetry. For the case $\mathcal{N} = 4$ we are interested in, we will show later that it is not $U(4)$ but $SU(4)$.

2.3.1 Representations

In analogue to what we did for the Poincare group, we want to find again the irreducible representations of the superalgebra. These representations are called supermultiplets, since a given irreducible representation contains various representations of the Lorentz algebra. This is reasonable since the Lorentz algebra is a subalgebra of the superalgebra. Before showing how we can construct these representations, we will state some important facts about them.

Firstly, since P^2 remains a Casimir operator for the superalgebra, the mass of all states in a multiplet is the same. Secondly, in each representation there is an equal number of bosonic and fermionic degrees of freedom. We can see that, by studying the operator: $(-1)^F$, which is defined by its action on the bosonic and fermionic states:

$$(-)^F|b>=|b> \qquad (-)^F|f>=-|b> \tag{2.17}$$

Since Q turns a fermionic state to a bosonic one and vice versa we will have: $(-1)^F Q = -Q(-1)^F$. Taking the trace of $(-)^F\{Q_\alpha, \bar{Q}_{\dot\alpha}\}$ we get:

$$\mathrm{Tr}[(-)^F\{Q_\alpha, \bar{Q}_{\dot\beta}\}] = \mathrm{Tr}[(-)^F(Q_\alpha\bar{Q}_{\dot\beta} - \bar{Q}_{\dot\beta}Q_\alpha)] = \tag{2.18}$$

$$= \mathrm{Tr}[-Q_\alpha(-)^F\bar{Q}_{\dot\beta} + Q_\alpha(-)^F\bar{Q}_{\dot\beta}] = 0 = 2\sigma^\mu_{\alpha\dot\beta}\mathrm{Tr}[(-)^F P_\mu] \tag{2.19}$$

Now, since $\mathrm{Tr}[(-)^F]$ is the difference between the number of bosons and fermions in a given supermultiplet we see that for fixed non-zero momentum $\mathrm{Tr}[(-)^F] = n_B - n_F = 0$.

Massless supermultiplets Following the same procedure we followed for the Poincare group, we go to the frame $p^\mu = (E, 0, 0, E)$. The states will be again labelled by the momentum p^μ and the helicity λ. In this frame from 2.10 we get:

$$\{Q^a_\alpha, \bar{Q}_{b\dot\beta}\} = 2\delta^a_b\sigma^\mu_{\alpha\dot\beta}P_\mu = 2\delta^a_b E(-\sigma^0 + \sigma^3)_{\alpha\dot\beta} = 4\delta^a_b E \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{2.20}$$

We see therefore that for $\alpha = \dot{\beta} = 2$:

$$\{Q_2^a, \dot{Q}_2^b\} = 0 \tag{2.21}$$

and thus if we act on a state $|p^\mu, \lambda\rangle$ we see that $Q_2^a|p^\mu, \lambda\,\rangle = 0 = \bar{Q}_2^a|p^\mu, \lambda\,\rangle =$ for all a. For the second component of the spinors we get from 2.20 $\{Q_1, \bar{Q}_2\} = 4E$. This indicates that we can define creation and annihilation operators:

$$a^b = \frac{Q_2^b}{2\sqrt{E}}, \qquad a_b^\dagger = \frac{\bar{Q}_{b\dot{1}}}{2\sqrt{E}} \tag{2.22}$$

so that they satisfy the algebra of fermionic creation and annihilation operators:

$$\{a^b, a_c^\dagger\} = \delta_c^b \qquad \{a^b, a^c\} = \{a_b^\dagger, a_c^\dagger\} = 0 \tag{2.23}$$

Let us first focus on the case of simple supersymmetry. Then we can build the representation by starting from a state for which $J_{12}|p^\mu, \lambda\,\rangle = \lambda|p^\mu, \lambda\,\rangle$ and which satisfies: $a|p^\mu, \lambda\,\rangle = 0$. We then act on it with $a^\dagger$. From the commutation relation $[J_{12}, \bar{Q}_{\dot{1}}] = \frac{1}{2}\bar{Q}_{\dot{1}}$ we see that $J_{12}\bar{Q}_{\dot{1}}|p^\mu, \lambda\,\rangle = (\lambda + \frac{1}{2})\bar{Q}_{\dot{1}}|p^\mu, \lambda\,\rangle$, which means that $a^\dagger$ raises the helicity by $\frac{1}{2}$. Because also the new state will be annihilated by $a^\dagger$, we conclude that the multiplet will consist of the two particle states:

$$|p^\mu, \lambda\,\rangle\,, \qquad |p^\mu, \lambda + \frac{1}{2}\,\rangle \tag{2.24}$$

If we want to have CPT invariance and since parity changes the sign of the helicity, we have to add to the above multiplet its CPT conjugate:

$$|p^\mu, -\lambda\,\rangle\,, \qquad |p^\mu, -\lambda - \frac{1}{2}\,\rangle \tag{2.25}$$

Some examples of supermultiplets that we will refer to in the next section are the scalar multiplet, which has two states with $\lambda = 0$ and the pair $\lambda = \pm\frac{1}{2}$ and corresponds to a complex scalar and a chiral fermion, and the vector multiplet, which is constructed from $\lambda = \frac{1}{2}$ and has the degrees of freedom of a gauge boson and a chiral fermion.

If we consider the case of extended supersymmetry, we have N creation operators and starting with a given state, we can construct 2^N states.

Massive Supermultiplets Let us consider now massive representations. Again, we go to the reference frame $p^\mu = (m, 0, 0, 0)$. Particle states will be again represented by $|p^\mu, s, s_3 >$. Now, the anticommutation relation between the charges is

$$\{Q_\alpha^a, \bar{Q}_{b\dot{\beta}}\} = 2m\delta_b^a(\sigma_0)_{\alpha\dot{\beta}} = 2m\delta_b^a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{2.26}$$

We have twice the number of creation and annihilation operators:

$$a_\alpha^b = \frac{Q_\alpha^b}{\sqrt{2m}}, \qquad (a^\dagger)_{\dot{\alpha}}^a = \frac{\bar{Q}_{\dot{\alpha}}^a}{\sqrt{2m}} \tag{2.27}$$

and so starting from a state with a given spin, we can create 2^{2N} different states.

2.3.2 Superfields

Our goal is to find Lagrangians that are invariant under supersymmetry. We , therefore, have to consider how supersymmetry is represented on the fields that create the particles. We will present here the $\mathcal{N}=1$ case, in order to illustrate how supersymmetry works. From the construction of the supermultiplets we can infer that there will be both bosonic and fermionic fields, and that the variation of a scalar field will give rise to a spinor field and vice versa. A very detailed constructive approach, starting from the variations of the fields, proceeding to the action of free fields and then introducing the interactions can be found in [3]. We will instead present a different method to find the supersymmetric action corresponding to a particular supermultiplet, which is much more easier and elegant.

In ordinary quantum field theories, we have fields defined on Minkowskian spacetime and an action that is invariant under Poincare transformations. Now, we can consider a generalization of normal fields to superfields, which in addition to the spacetime coordinates depend also on fermionic coordinates θ_α and $\bar{\theta}_{\dot{\alpha}}$. This new space is called superspace, is denoted by $\mathcal{R}^{4|4}$ and its coordinates are: $z^A=(x^\mu,\theta_\alpha,\bar{\theta}_{\dot{\alpha}})$. These superfields contain now the various fields that are mapped to each other under the supersymmetry transformations.
A generic element of the supersymmetry group will be written as:

$$G(x,\theta,\bar{\theta})=e^{-ix_\mu P^\mu+i\theta^\alpha Q_\alpha+i\bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}} \tag{2.28}$$

This can be used to find how supercoordinates are changed under a supersymmetry transformation:

$$(x^\mu,\theta,\bar{\theta})\rightarrow(x^\mu+i\theta\sigma^\mu\bar{\epsilon}-i\epsilon\sigma^\mu\bar{\theta},\theta+\epsilon,\bar{\theta}+\bar{\epsilon}) \tag{2.29}$$

As for the usual fields, the action of the group on the superfields will be defined through a differential representation of the generators. So, one can find (see [1] for example) that the charges Q and $\bar{Q}$ are represented by the following operators:

$$Q_\alpha=\frac{\partial}{\partial\theta^\alpha}-i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu, \tag{2.30}$$

$$\bar{Q}_{\dot{\alpha}}=\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}}-i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \tag{2.31}$$

The most general superfield can be written as:

$$F(x,\theta,\bar{\theta})=f^1(x)+\theta f^2(x)+\bar{\theta}\bar{f}^3(x)+\theta^2f^4(x)+\bar{\theta}^2f^5(x)+\theta\sigma^\mu\bar{\theta}f^6_\mu+\theta^2\bar{\theta}\bar{f}^7+\bar{\theta}^2\theta f^8(x)+\theta^2\bar{\theta}^2f^9(x) \tag{2.32}$$

where $f^1(x),f^4(x),f^(5),f^9(x)$ are scalars,$f^2(x),f^8(x)$ are left-handed Weyl spinors, $f^3(x),f^7(x)$ are right-handed Weyl spinors and $f^6(x)$ is a vector field. We see that we have an equal amount of bosonic and fermionic degrees of freedom, so the superfield forms a representation of the superalgebra. The transformation of the component fields can be derived by acting on the superfield F with the operators Q and $\bar{Q}$:

$$\delta_{\epsilon,\bar{\epsilon}} F = (\epsilon Q + \bar{\epsilon}\bar{Q})F \tag{2.33}$$

The variation of the superfield is of course a superfield itself, and thus by comparing terms with the same power of the fermionic coordinates we can get the variation of the component fields.

These representations are in general reducible. In the particular case of $\mathcal{N} = 1$ they are always reducible since due to the larger number of degrees of freedom, the component fields do not fit into the $N = 1$ supermultiplets we studied in the previous section. Therefore, we will have to put constraints on the superfields.

Chiral superfield It is important to find conditions, that will lead to superfields, which under a supersymmetry transformation remain of the same type. The first example we will study is the chiral superfield, which is determined by the condition:

$$\bar{D}_{\dot{\alpha}} \Phi(x, \theta, \bar{\theta}) = 0 \tag{2.34}$$

where $\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial \theta^{\dot{\alpha}}} - i\theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu}$. This constraint leads to:

$$\Phi(x, \theta, \bar{\theta}) = \phi(x) + i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}\phi(x) + \frac{1}{4}\theta^2\bar{\theta}^2\partial_{\rho}\partial^{\rho}\phi(x) + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta^2\partial_{\mu}\psi(x)\sigma^{\mu}\bar{\theta} + \theta^2 F(x) \tag{2.35}$$

We see that we have a complex scalar field ϕ, a left-handed Weyl spinor ψ and an auxiliary complex scalar F, which helps balance the degrees of freedom off-shell. Remember a complex scalar has two bosonic degrees of freedom and a Weyl spinor has 4 off-shell. Thus, we need an auxiliary complex scalar field. However, when we will construct an action for the chiral superfield, F will turn out to be non-dynamical, as it should. To sum up we see that we got a superfield that corresponds to the chiral multiplet.

The next step is to find an action that will describe this multiplet. In general we want to find Lagrangians that are invariant under a supersymmetry transformation up to a total derivative. If we compute the variations in 2.33 we see that the only variation that is a total derivative is that of the $\theta^2\bar{\theta}^2$ term:

$$\delta_{\epsilon,\bar{\epsilon}} f^9 = \frac{i}{2}\epsilon\sigma^{\mu}\partial_{\mu}\bar{f}^7 - \frac{i}{2}\partial_{\mu} f^8 \sigma^{\mu}\bar{\epsilon} \tag{2.36}$$

So, using the usual laws for integration of Grassmann variables we could define an action:

$$S = \int d^4x d^2\theta d^2\bar{\theta} F \tag{2.37}$$

which is automatically invariant under supersymmetry, since the integration keeps only the component f^9. It turns out due to dimensional reasons that for chiral superfields the only possibility for $F(x, \theta, \bar{\theta})$ is[1] $F = \Phi^{\dagger}\Phi$ and the Lagrangian is therefore:

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} = \int d^4x(\partial^{\mu}\phi\partial_{\mu}\phi^* - i\bar{\psi}\bar{\sigma}^{\mu}\partial_{\mu}\psi + ff^*) \tag{2.38}$$

[1]The antichiral field is similarly defined as the superfield satisfying $D_{\alpha}\Phi^{\dagger} = \left(-\frac{\partial}{\partial\theta^{\alpha}} + i\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_{\mu}\right)\Phi^{\dagger} = 0$

We should note here that the superfield $W = -\frac{m}{2}\Phi^2 + \frac{g}{2}\Phi^3$ has also a component which transforms as a total derivative (specifically its θ^2 term) and so we could add to our Lagrangian a corresponding term.

Vector Superfield The next superfield we will consider is the real or vector superfield, which is obtained by the condition:

$$V(x,\theta,\bar{\theta}) = V^{\dagger}(x,\theta,\bar{\theta}) \tag{2.39}$$

This condition leads to

$$\begin{aligned}
V(x,\theta,\bar{\theta}) &= C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) + \frac{i}{2}\theta^2(M(x)+iN(x)) - \frac{i}{2}\theta^2(M(x)-iN(x)) - \\
&-\theta\sigma^{\mu}\bar{\theta}A_{\mu}(x) + i\theta\bar{\theta}^2\left(\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^{\mu}\partial_{\mu}\chi(x)\right) - i\bar{\theta}^2\theta\left(\lambda(x) + \frac{i}{2}\sigma^{\mu}\partial_{\mu}\chi(x)\right) + \\
&+\frac{1}{2}\theta^2\bar{\theta}^2\left(D(x) + +\frac{1}{2}\partial_{\rho}\partial^{\rho}C(x)\right)
\end{aligned} \tag{2.40}$$

Since, we have now the vector field A_{μ}, we expect gauge symmetry to appear.

Let us look at the kinetic term $K = \bar{\Phi}\Phi$. This is invariant under $\Phi \to e^{i\alpha}\Phi$ and $\bar{\Phi} \to e^{-i\alpha}\Phi^{\dagger}$. We can turn this global symmetry to a local one, by promoting α to a chiral field Λ. We see then that as we expected $K \to \bar{\Phi}e^{-i\bar{\Lambda}}e^{i\Lambda}\Phi$, and is therefore not invariant, but as in the non-supersymmetric case we have to introduce a connection. We introduce therefore a real superfield V, which transforms as

$$e^{V} \to e^{i\bar{\Lambda}}e^{V}e^{-i\Lambda} \tag{2.41}$$

and write the kinetic term as $\bar{\Phi}e^{V}\Phi$.

Abelian case For the simple case of an abelian superfield, the transformation law of the vector field simply becomes $V \to V - i(\Lambda - \bar{\Lambda})$. This can be used in order to eliminate some of the fields of the vector superfield, and leave only those that describe the degrees of freedom of the vector multiplet.

Before discussing the non-abelian case, which is of interest for us, we will briefly discuss Non-Abelian gauge theories.

Non-Abelian gauge theory Let us consider a set of fields ϕ^i which transform in a certain representation of a non-abelian Lie group. The representation might be the fundamental one, which for groups such as $U(N)$ or $SU(N)$ means that the the fields can be considered as elements of a N-dimensional vector space on which the group acts in the following way: $\phi^j(x) \to \phi^j(x) = U^j_k(x)\phi^k(x) = \left(e^{i\alpha^a(x)T_a}\right)^j_k\phi^k(x)$. Another representation which is the one we will have in the $\mathcal{N} = 4$ SYM is the adjoint one, where the field can be written as $\Phi = \Phi^a T_a$. So the field is now composed of matrices transforming as:

$$\Phi^i{}_j \equiv \Phi^a(T_a)^i_j \to \Phi^i{}_j = \left(e^{i\alpha^b T^b}\right)^i_k \Phi^a(T_a)^k_l\left(e^{-i\alpha^c T_c}\right)^l_j \tag{2.42}$$

The covariant derivative is $D_\mu \Phi = \partial_\mu \Phi + i[A_\mu, \Phi]$, the field strength is $F_{\mu\nu} = -i[D_\mu, D_\nu]$ and the kinetic term for the gauge field will be expressed as a trace of $F_{\mu\nu}F^{\mu\nu}$.

Non-abelian case In order to write an action for the vector multiplet degrees of freedom, V is not a good choice, since we would want to have a superfield that under the gauge group transforms as

$$\mathcal{V} \to e^{i\Lambda}\mathcal{V}e^{-i\Lambda} \tag{2.43}$$

in order to be able to build gauge invariant quantities by taking traces.
It turns out that such a field is

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}\left(e^{-V}D_\alpha e^V\right) \tag{2.44}$$

which contains also the field strength. The transformation property 2.43 means that the component fields are in the adjoint represention of the gauge group. At this point, we also need to mention that all fields in a given multiplet will transform in the same way under the action of the gauge group, since gauge symmetry commutes with supersymmetry. If this was not the case, then supersymmetry would have to be local too.

2.4 $\mathcal{N} = 4$ SYM

It's finally time to introduce the $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory. As the name suggests we have now $\mathcal{N} = 4$ independent supersymmetries. We want to restrict our theory to particles with spin ≤ 1, in order not to have gravity. This means that we have to consider only massless supermultiplets, since in that case there are only 4 creation operators, and so starting with helicity -1 we will have a multiplet with 1 state with $\lambda = -1$, four states with $\lambda = -\frac{1}{2}$, 6 six states with $\lambda = 0$, four states with $\lambda = \frac{1}{2}$ and one state with $\lambda = 1$. It is clear that we have the maximum allowed supersymmetry for a multiplet with $\lambda \leq 1$. This is why theories with $\mathcal{N} = 4$ in four spacetime dimensions are called maximally supersymmetric. The multiplet we constructed shows that we will have a gauge field, the degrees of freedom of which will correspond to the $\lambda \pm 1$ states, 6 real scalars corresponding to $\lambda = 0$ and 4 Weyl fermions corresponding to $\lambda = \pm\frac{1}{2}$. All fields transform in the adjoint representation of the SU(N) gauge group. As we already said, this means that we can write them as $\Phi^I = \Phi^I_a T^a$, $A_\mu = A^a_\mu T^a$, $\Psi = \Psi^a T^a$, where T^a are the generators of the gauge group, and are $N \times N$ traceless, Hermitian matrices satisfying the $su(N)$ lie algebra and are normalized as $\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$. The action of $\mathcal{N} = 4$ SYM is:

$$S = \int d^4x(-\frac{1}{4}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\mathrm{Tr}D_\mu\phi_I D^\mu\phi^I - \frac{g^2_{\mathrm{YM}}}{2}\sum_{I<J}\mathrm{Tr}[\phi_I, \phi_J]^2 + \tag{2.45}$$

$$+\mathrm{Tr}\bar{\psi}^a\sigma^\mu D_\mu\psi_a - \frac{ig_{\mathrm{YM}}}{2}\mathrm{Tr}\sigma^{ab}_i\psi_a[\Phi^I, \psi_b] - \frac{ig_{\mathrm{YM}}}{2}\mathrm{Tr}\sigma^i_{ab}\bar{\psi}[\Phi_I, \bar{\psi}^b]) \tag{2.46}$$

As we discussed in a previous section, there is also a global R-symmetry $SU(4)$, which rotates the 6 scalars to one another. From the previous section we would expect that this symmetry is $U(4)$, but it is $SU(4)$. This happens because, if we write $U(4) = U(1) \times SU(4)$, we see that the generator of the $U(1)$ in four spacetime dimensions commutes with the whole algebra and so is trivially realized (in other words is zero).

Before discussing about symmetries again, it would be useful to discuss why one studies this theory. Firstly, it has a large amount of symmetry which makes possible the analytical calculation of many physical quantities. An obvious reason is also that it is used in the context of the AdS/CFT correspondence and specifically the first example of such a duality was between N=4 SYM and type IIB string theory on $AdS_5 \times S^5$. Perhaps, the most important aspect is that $N = 4$ SYM is conformally invariant and retains this property also in the quantum level, since it has been shown that the β function is zero to all orders in perturbation theory.

2.5 Superconformal algebra

In this section we will combine the supersymmetry of $\mathcal{N} = 4$ with the conformal symmetry which also carries, in order to get a superconformal algebra. As we already said the complete symmetry group of $\mathcal{N} = 4$ SYM is $PSU(2, 2|4)$ and its corresponding algebra is $\mathfrak{psu}(2, 2|4)$. So far, we saw the bosonic generators of the conformal group $J_{\mu\nu}, P_\mu, K_\mu$ and D and the supercharges Q^a_α and $\bar{Q}^a_{\dot\alpha}$, which are the superpartners of P_μ. Now, in order to ensure closure of the algebra we have to consider further supercharges S^a_α and $S^a_{\dot\alpha}$ which are the superpartners of K_μ. The whole superconformal algebra can be found for example in [1]. Here we will write some commutation relations of importance:

$$[Q^a{}_\alpha, K^\mu] = i\sigma^\mu_{\alpha\dot\alpha}\bar{S}^{a\dot\alpha}, \quad [\bar{Q}_{a\dot\alpha}, K^\mu] = -\epsilon_{\dot\alpha\dot\beta}(\bar\sigma^\mu)^{\dot\beta\alpha}S_{a\alpha} \tag{2.47}$$

$$\{S_{a\alpha}, \bar{S}^b_{\dot\beta}\} = 2\sigma^\mu{}_{\alpha\dot\beta}K_\mu\delta^b_a, \ \{S_{a\alpha}, S_{b\beta}\} = \{\bar{S}^a_{\dot\alpha}, \bar{S}^b_{\dot\beta}\} = 0, \ [S_{a\alpha}, K_\mu] = [\bar{S}^a_{\dot\alpha}, K^\mu] = 0 \tag{2.48}$$

$$\{Q^a_\alpha, S_{\beta b}\} = \epsilon_{\alpha\beta}(\delta^a_b D + R^a_b) + \frac{1}{2}\delta^a_b J_{\mu\nu}(\sigma^{\mu\nu})_{\alpha\beta} \tag{2.49}$$

where R^b_a are the R-symmetry generators.

2.5.1 Representations of the superconformal algebra

We consider again local operators $\mathcal{O}(x)$ composed by the elementary fields of our theory. As usual these operators can be characterized by their dimension Δ and their spin $\mathcal{J}_{\mu\nu}$:

$$[D, \mathcal{O}(0)] = -i\Delta\mathcal{O}(0), \qquad [J_{\mu\nu}, \mathcal{O}] = -\mathcal{J}_{\mu\nu}\mathcal{O}(0) \tag{2.50}$$

Similarly with what we did with the generators K_μ and P_μ, we can get from the commutation relations that S^a_α lowers the conformal dimension of an operator $\mathcal{O}(0)$ by 1/2. Therefore, due to unitarity, there is again a lower bound and there are some operators that satisfy:

$$[S^a_\alpha, \mathcal{O}(0)] = [\bar{S}_{\dot\alpha a}, \mathcal{O}(0)] = 0 \tag{2.51}$$

for all $\alpha, \dot\alpha, a$. These operators are called superconformal primaries and have the lowest dimension in a given supermultiplet. As the name suggests, they are alco conformal primaries, but the inverse is not always true. A superconformal operator along with its descendants, which are constructed by acting on the superconformal primary with the generators, makes up an irreducible representation of $\mathfrak{psu}(2, 2|4)$. In fact, the superconformal primaries are in one to one correspondence with the irreducible representations of $\mathfrak{psu}(2,2|4)$. Since the group is non-compact these representations are infinite dimensional. For reasons that will appear shortly, we will be interested in a subset of these superconformal primaries, called chiral primaries. These operators in addition to 2.51 satisfy also:

$$[Q^a_\alpha, \mathcal{O}(0)] = 0 \tag{2.52}$$

for at least one α, a. So, the representations which correspond to these operators, are "smaller", though still infinite of course. The R-symmetry group $SU(4) \simeq SO(6)$ has three generators that commute with all its other generators. This means that we can classify the operators in addition to their spin and conformal dimension with three numbers related to the R-symmetry group, the R-charges. It turns out[5] that we need only consider operators with $(J, 0, 0)$ R-numbers and that for chiral primaries the conformal dimension equals the number J. We can now show that the reason why we are interested in these operators is that they have anomalous dimension zero. In general, when we go to the quantum theory the dimension of the operators will get corrected by a quantity called anomalous dimension, which depends on the coupling g_{YM}. Within a given supermultiplet, all operators have the same anomalous dimension, since the generators can change the dimension by 1/2. An example can help us understand this better. If an operator has bare dimension Δ_0 and another one has bare dimension $\Delta_0 + \frac{1}{2}$, the anomalous dimension they get cannot be different, since these two operators are connected by the action of S^a_α, which raises the dimension by 1/2. Morover, by studying 2.49

$$0 = [\{S, Q\}, \mathcal{O}(x)] = [J + D + R, \mathcal{O}(0)] \sim (\Delta + R + \mathcal{J})\mathcal{O} \tag{2.53}$$

we see that the conformal dimension is a function of the spin and the R-charges. The spin certainly does not change by varying the coupling, while the same holds for the R-symmetry charges, since they are quantized numbers.

2.6 Gauge invariant operators

Moving on to our theory, we should only consider gauge invariant operators, since these correspond to physical observables. As, we showed earlier, the fields will transform in the adjoint representation of the gauge group, which means that:

$$\mathcal{W} \to U \mathcal{W} U^{-1} \tag{2.54}$$

where by $\mathcal{W}$ we collectively refer to the fields. So, it clear that in order to have gauge invariant operators we have to consider traces of the elementary fields, since then under a gauge transformation using the cyclicity of the trace we can easily see that these are gauge invariant. More specifically, we will consider single trace operators, which are operators of the form:

$$\mathcal{O}(x) = \mathrm{Tr}(\chi_1(x)\chi_2(x)...\chi_L(x)) \tag{2.55}$$

where $\chi_I(x)$ can be any of the component fields of $\mathcal{N} = 4$ SYM. We could also consider operators which are products of traces but it turns out that in the large N limit, all information is in the single trace operators.

2.7 Planar limit

In 1974, 't Hooft noticed that non-abelian gauge theories become considerably simplier if we take the number of colors N to infinity[2]. This limit is particularly useful in the context of the AdS/CFT correspondence. To illustrate its relevance to our paper we consider a toy model resembling Yang-Mills theory, with only a single scalar field ϕ. In particular we consider the field transforming in the adjoint representation of the gauge group $SU(N)$:

$$\phi = \phi^a T_a \to \phi^i_j = \phi^a (T_a)^i_j \tag{2.56}$$

and we have a Lagrangian with a quartic vertex proportional to the coupling constant square g^2 and a cubic vertex proportional to g:

$$\mathcal{L} = -\frac{1}{2} Tr(\partial_\mu \phi \partial^\mu \phi) + gTr(\phi^3) + g^2 Tr(\phi^4) = \tag{2.57}$$

$$\mathcal{L} = \frac{1}{g^2}\left[-\frac{1}{2} Tr(\partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi}) + Tr(\phi^3) + Tr(\tilde{\phi}^4)\right] \tag{2.58}$$

where in the second line we rescaled our field: $\tilde{\phi} = g\phi$.

A consistent way to take the $N \to \infty$ limit, is to define the 't Hooft coupling $\lambda \equiv g^2 N$ which is to remain constant when $N \to \infty$. From 2.58 it seems that this limit is not well defined, but we have to keep in mind that the number of components of the fields goes to ∞ as well, and it turns out that these two divergences cancel out.

As we can see from the Lagrangian each vertex contributes a factor of $1/g^2 = N/\lambda$, a propagator being the inverse of the kinetic term scales as $g^2 = \lambda/N$, while each loop gives a factor of N which comes from the sum over the color indices. Therefore, a Feynman diagram with V vertices, E propagators and L loops is proportional to:

$$N^{V-E+L}\lambda^{E-V} = N^{\chi}\lambda^{E-V} = N^{2-2g}\lambda^{E-V} \tag{2.59}$$

where χ is the Euler characteristic and g is the genus, where these quantities make sense if we consider the Feynman diagrams as surfaces.
The propagator of the field ϕ is:

$$\left\langle \tilde{\phi}^i_j(x)\tilde{\phi}^k_l(y)\right\rangle = \delta^i_l \delta^k_j \frac{g^2}{4\pi^2(x-y)^2} \tag{2.60}$$

This can be seen using the completeness relation the generators satisfy[3] $\sum_{a=1}^{N^2}(T_a)^i_j(T_a)^k_l = \delta^i_l\delta^k_j$. Equation 2.60 indicates that we can use the so-called double line notation.

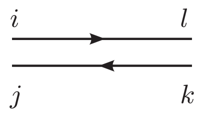

Figure 2: Double line notation

Two characteristic examples of feynman diagrams are shown in Figure 3. The first

[2]In QCD, where we have the gauge group SU(3), we say that there are 3 colors. Similarly, in a SU(N) gauge theory we say that we have N colours.

[3]For the feynman rules for NxN matrix fields see for example Srednicki chapter 80

(a) (b)

Figure 3

one has $V = 2, E = 3, L = 3$ and the second one $V = 4, E = 6, F = 2$. To understand why we have these numbers of closed loops, just follow the arrows. The left diagram has genus zero and scales as N^2, while the right one has genus one and scales as N^0. These examples help us understand that physical quantities can be expressed as expansions in the genus. Take for example the generating functional:

$$iW = \sum_{g=0}^{\infty} N^{2-2g} \sum_{l=0}^{\infty} c_{g,j}\lambda^j = \sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda) \tag{2.61}$$

We observe, therefore, that in the large N limit, diagrams with genus zero will contribute much more that the rest. These diagrams are called planar diagrams and can be drawn in the plane without any lines crossing.

3 One-loop anomalous dimension for single trace scalar operators

At this section, we are going from tree level to one loop diagrams. Through this, we will calculate the corresponding anomalous dimension. In particular, we start from the general expression of the two point function:

$$\langle \mathcal{O}(x)\overline{\mathcal{O}}(y)\rangle \approx \frac{1}{|x-y|^{2\Delta}} \tag{3.1}$$

We denoted $\Delta = \Delta_0 + \gamma$, where Δ_0 is the bare dimension of the field and γ is the anomalous dimension. The approximation symbol comes from the fact that we consider up to a specific quantum correction order. Renormalizing 3.1, the general form of the two point function becomes:

$$\langle \mathcal{O}(x)\overline{\mathcal{O}}(y)\rangle \approx \frac{1}{|x-y|^{2\Delta_0}}(1 - \gamma ln\big(\Lambda^2|x-y|^2\big)) \tag{3.2}$$

We set a UV cut-off Λ and we see that we have a logarithmic divergene, which is typical for one-loop corrections. So, the next task is to calculate the explicit relation for the anomalous dimension. Before going there, we must set the rules of the game, that is how to calculate tree level diagrams.

We will start from chiral primaries and then generalize to arbitrary gauge invariant operators. Also we need to take the planar limit, that is $N \to \infty$, so that computations become considerably easier-as it will be shown shortly. As an example of a chiral primary we can consider:

$$\Psi_L = \frac{(4\pi)^{L/2}}{\sqrt{L}N^{L/2}} Tr Z^L = \frac{(4\pi)^{L/2}}{\sqrt{L}N^{L/2}} Z^A_B Z^B_C ... Z^{\cdot\cdot}_A \tag{3.3}$$

where $Z = \frac{1}{\sqrt{2}}(\phi^1 + i\phi^2)$ is a complex scalar field, composed by two of the real fields of $\mathcal{N} = 4$ SYM. Since, the fields are represented by $N \times N$ traceless matrices, the non-trivial chiral primaries have $L \geq 2$. When we have a general 2-point-function between a $\Psi_L(x)$ and $\overline{\Psi}_L(y)$, we start by Wick contracting the different Z's and using:

$$\langle Z^A_B(x)\overline{Z^C_D}(y)\rangle = \frac{\delta^A_D \delta^C_B}{4\pi^2|x-y|^2} \tag{3.4}$$

Thus, the procedure of calculating the two-point function of two chiral primaries isn't difficult conceptually. But as we can have different combinations in contractions and probably $L \gg 1$, we understand that we will end up with a pretty nasty calculation. However, having considered the planar limit, this calculation turns out to be much easier. We will sketch out the calculation for $L = 3$ and we will see that this is easily generalized to an arbitrary L. So, we have essentially a calculation of the following form:

$$\langle Z^A_B Z^B_C \underline{Z^C_A(x)\overline{Z}^{A'}_{C'}} \overline{Z}^{C'}_{B'} \overline{Z}^{B'}_{A'}(y)\rangle \tag{3.5}$$

Moreover, we choose to contract indices as indicated by the underline in the expression. Using 3.4, considering $\delta^a_a = N$, it's easy to see that this gives N^3. Although apart from this, another choice could be to contract the 3rd with the 5th, the 2nd with the 6th etc. This will give also N^3. We have totally 3 different choices that give this result,But there are also other kind of contractions. Their contribution is however is N. And as we have chosen the planar limit, these contributions are considered subleading. The graphs that correspond to the contractions we descibed are called planar graphs.

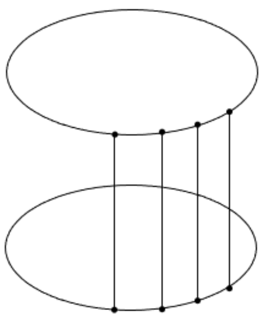

Figure 4: Planar

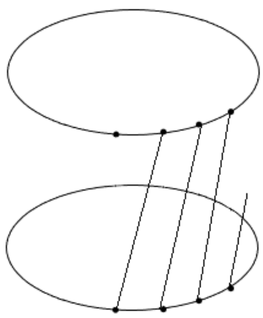

Figure 5: Planar

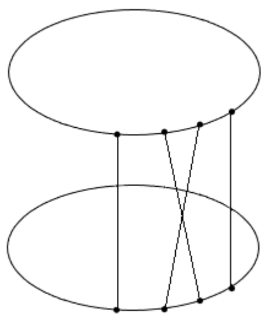

Figure 6: Non planar

It is not difficult to generalize this result to L different sites. Again there will be L different contractions which will give N^L. Actually the argument for planar graphs stands only when $L \ll N$. But as we take large N limit, this isn't so restrictive. Bearing all these in mind and using the general expression for the chiral primaries(with the normalization constants) we get:

$$\langle \Psi_L(x)\overline{\Psi_L}(y)\rangle = \frac{LN^L}{(\sqrt{L}N^{L/2})^2|x-y|^{2L}} = \frac{1}{|x-y|^{2L}} \tag{3.6}$$

Up until now, we considered only chiral primaries. But we can as well generalize for any scalar field ($C_{I_1,\dots,I_L}$ is a symmetry factor):

$$\mathcal{O}_{I_1,I_2,..,I_L}(x)=\frac{((4\pi)^{L/2})}{\sqrt{C_{I_1,\dots,I_L}}N^{L/2}}Tr\big(\phi_{I_1}\phi_{I_2}...\phi_{I_k}\big) \tag{3.7}$$

So if we go to calculate the -tree level- point function as for the color indices the procedure is the same. But we need to consider the additional index I_i (let's call them flavours, in analogy with QCD!). Contracting these fields, by demanding to have the same kind of field for non-zero contribution:

$$\langle\mathcal{O}_{I_1,I_2,..,I_L}(x)\overline{\mathcal{O}}^{J_1,\dots,J_k}(y)\rangle_{tree}=\frac{1}{C_{I_1,\dots,I_L}}\big(\delta^{J_1}_{I_1}\delta^{J_2}_{I_2}...+cycles\big)\frac{1}{|x-y|^{2L}} \tag{3.8}$$

The cycles in this expression come from the fact that we can contract the indices in different ways, but again in order to keep only the planar graphs, we can take all the cyclic permutations between the I_i and J_j.

Having explained the correlator for the tree level, we now go on to one loop. The tree level correlators can be extracted just by using the symmetries. But for the corrections, we need to take a look at the action. We will show explicitly how the scalar terms of the action contribute to the anomalous dimension, even though other terms contribute as well. After having this result, we will explain how this result is applicable to the rest of the terms. We consider, therefore, :

$$S_{bos}=\frac{g^2_{YM}}{2}\int d^4x\Big\{\frac{1}{2}Tr\mathcal{F}+Tr\mathcal{D}_\mu\phi_I\mathcal{D}^\mu\phi^I-\frac{1}{2}\sum_{I<J}Tr[\phi_I,\phi_J]^2\Big\} \tag{3.9}$$

We see that we have 2 kind of interactions. The first comes from the interaction between the scalars and the gauge boson (from the covariant derivative term) and the second is the interaction between the 4 scalar fields. At this work, we are interested only in the latter. We will comment later for the other term. So the last term is:

$$\frac{g^2_{YM}}{4}\int\Big[d^4x\sum_{I,J}Tr(\phi_I\phi_I\phi_J\phi_J)-Tr(\phi_I\phi_J\phi_I\phi_J)\Big] \tag{3.10}$$

Doing this, we added an interaction vertex in terms of diagrams. So we need to calculate the action in the correlator. To do so, we need to take the S-matrix. We need to do again the contractions etc. As we have 4 fields in the interaction vertex, we should contract them with 2 ingoing and 2 outgoing fields. Moreover, as we take the planar limit, both the two ingoing and two outgoing fields must be neighbouring (if that's not the case, the contractions will necessarily lead to non-planar graphs, see Figure 7). So, the subcorrelator is (we refer to subcorrelator, as we need to do the same procedure for every field/site of the state):

$$\langle((\phi_{I_k}\phi_{I_{k+1}})^A_C(x))\left(\frac{ig^2_{YM}}{4}\int d^4z\sum_{I,J}(Tr(\phi_I\phi_I\phi_J\phi_J)(z)-Tr(\phi_I\phi_J\phi_I\phi_J)(z))\right)(\phi^{J_k}\phi^{J_{k+1}})^{C'}_{A'}(y)\rangle \tag{3.11}$$

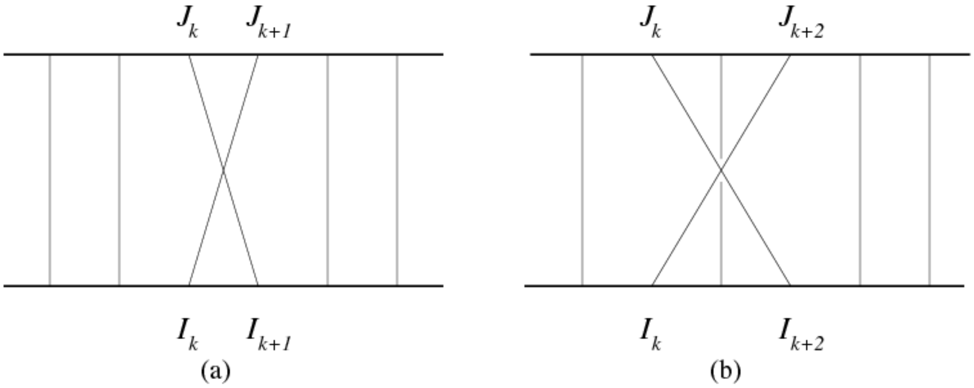

Figure 7: Diagrams with the introduction of the quarctic interaction. The graph (a) describes a planar diagram, while graph (b)depicts a non-planar interacting diagram

We have two terms in the interaction vertex. These need to be contracted with the state fields. Due to the planar limit, the contractions must be with subsequent fields. For instance, we can contract $\phi_{I_{k+1}}$ with the second field but this means that ϕ_{I_k} must go with the third one. So we understand that the first term in the interaction vertex has two choices. Either relates the incoming with the outcoming states or contracts an incoming with an incoming and an outcoming with an outcoming. On the other hand, the second term can only relate incoming to outgoing ones. Therefore (3.11) becomes

$$i\frac{N}{(4\pi^2)^2}\delta^A_{A'}\delta^C_{C'}\frac{g^2_{YM}N}{64\pi^4}\left(2\delta^{J_k}_{I_k}\delta^{J_{k+1}}_{I_{k+1}}+2\delta_{I_k,Ik+1}\delta^{J_k,J_{k+1}}-4\delta^{J_{k+1}}_{I_k}\delta^{J_k}_{I_{k+1}}\right)\int\frac{d^4z}{|z-x|^4|z-y|^4} \tag{3.12}$$

The additional factor N came from the color summation in the interaction vertex. Next step is to calculate the integral of z. This is essentially UV divergent for $z \to x, y$. Note that someone usually calculates loop integrals in momentum space and the divergence becomes evident there. Now we are working in the position space and the UV divergence arises when z approaches the singular points. But in our case, it is convenient working on the position space, as we have defined all of the fields in specific positions (local fields). A common practice we do to combat UV divergences is to set some UV limit Λ. So we will have a low limit at Λ^{-1}, as we are in the position space. Another step is to do a Wick rotation ($dz \to idz_E$). If we want to express the cut-off condition, it is $|z_E - x| \geq \Lambda^{-1}, |z_E - y| \geq \Lambda^{-1}$ According to what we told about the divergence, it makes sense to consider the main contribution to the integral to come from the regions near the singular points. So based on this claim, we are going to perform a nasty trick (admittedly!) to get an approximation of the result. Consider $\xi = |z - x|$ and $\xi \approx 0$ (z is near x). This will mean that $|z - y| \approx |x - y|$ (As our initial consideration was that z and x are almost coincident). We need to define the new limits of the integral. So we said that we have a lower cut-off at Λ^{-1}. In order to find the upper limit we return again to the claim that the highest contribution comes near the two points x,y. We took as starting point somewhere near x. So the alleged infinity for z will be at z_∞=y.

Therefore $\xi_{up} = |y-x|$. The integral becomes:

$$i \int \frac{d^4 z_E}{|z-x|^4 |z-y|^4} \approx \frac{2i}{|x-y|^4} \int_{\Lambda^{-1}}^{|x-y|} \frac{d\xi \Omega_3}{\xi} = \frac{\pi^2 i}{|x-y|^4} ln\Big(\Lambda^2 |x-y|^2\Big) \tag{3.13}$$

So the final result for the subcorrelator is:

$$-\frac{N\delta^A_{A'}\delta^C_{C'}}{(4\pi^2)^2|x-y|^4}\frac{\lambda}{16\pi^2}\left(\delta^{J_k}_{I_k}\delta^{J_{k+1}}_{I_{k+1}} + \delta_{I_k,Ik+1}\delta^{J_k,J_{k+1}} - 2\delta^{J_{k+1}}_{I_k}\delta^{J_k}_{I_{k+1}}\right) \tag{3.14}$$

Up to now, we were interested in the interaction term between the 4 scalars. But there are also the other term that couple scalar with the gauge field. Given that we consider the external states (ingoing and outgoing) consisting solely by scalar fields. Moreover, we can have other terms coming either from the gluon exchange (again an analogy from QCD, we are talking for the gauge boson) or from fermion loops. As the gluon carries no R-charge and the fermion loops are essentially self-energy diagrams, we understand that the external states-compared to the tree level- won't change at all and every change will occur internally. Then we can assign for these diagrams an additional parameter that bears resemblance with the first one of the previous result. We define also a constant C, which we will determine shortly. These diagrams are:

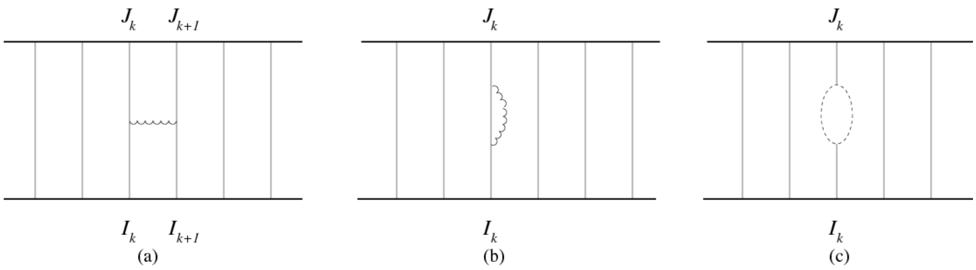

Figure 8: Planar graphs originated from the coupling of scalar and gauge field. The exterior flavor structure remains the same. (a) gluon exchange between neighbouring fields (b)Scalar self energy from a gluon (c) Scalar self energy from a fermion loop

Generalizing for the whole correlator we need to use what we extracted before. Moreover, we need to perform the summation for the whole array of operators. Finally we add an additional term for the cycles. This comes from the fact that we can have L planar diagrams, as we can contract a site with L different for the conjugate (but as soon as we determine the first contraction, the other fields have no choice). In total:

$$\langle \mathcal{O}_{I_1,..,I_L}(x)\overline{\mathcal{O}}^{J_1,...,J_L}(y)\rangle_{O.L.} = \frac{\lambda}{16\pi^2}\frac{ln\big(\Lambda^2|x-y|^2\big)}{|x-y|^{2L}}\times$$
$$\times \sum_{l=1}^{L}(2P_{l,l+1} - K_{l,l+1} - 1 + C)\frac{1}{\sqrt{C_{I_1}..C_{I_L}C_{J_1}..C_{J_L}}}\delta^{J_1}_{I_1}\delta^{J_2}_{I_2}...\delta^{J_L}_{I_L} + cycles$$

What we did now is nothing but rearranging and relabelling of what we have written up to now. So the new terms we introduced have the following properties:

$$P_{l,l+1}\delta^{J_1}_{I_1}\delta^{J_2}_{I_2}...\delta^{J_L}_{I_L} = \delta^{J_1}_{I_1}...\delta {I_l}^{J_{l+1}}\delta^{J_l}_{I_{l+1}}...\delta^{J_L}_{I_L} \tag{3.15}$$

$$K_{l,l+1}\delta^{J_1}_{I_1}\delta^{J_2}_{I_2}...\delta^{J_L}_{I_L} = \delta^{J_1}_{I_1}...\delta_{I_l,I_{l+1}}\delta^{J_l,J_{l+1}}...\delta^{J_L}_{I_L} \tag{3.16}$$

Combining the tree level and one loop result we aqcuired:

$$\langle \mathcal{O}_{I_1,.,I_L}(x)\overline{\mathcal{O}}^{J_1,.,J_L}(y)\rangle = \frac{1}{|x-y|^{2L}}\left(1-\frac{\lambda ln\left(\Lambda^2|x-y|^2\right)}{16\pi^2}\sum_{l=1}^{L}\left(2P_{l,l+1}-K_{l,l+1}-1+C\right)\right)\delta_{I_1}^{J_1}..\delta_{I_L}^{J_L}+ccl \tag{3.17}$$

Comparing this result with the initial general expansion of the correlator, we can understand that the second term is the anomalous dimension. If we want to express it as an operator in analogy with the dilatation operator:

$$\Gamma = \frac{\lambda}{16\pi^2}\sum_{l=1}^{L}\left(2P_{l,l+1}-K_{l,l+1}-1+C\right) \tag{3.18}$$

We need to define the value of C. To do so, we will return to the chiral primary, the case we studied in the beginning of this section. Its simplicity and the fact that they don't have anomalous dimension is extremely helpful and make our calculation simple. Thus the chiral primary has only Z fields in its trace and therefore any interchange of Z gives essentially the same result, $P_{l,l+1}\Psi_L = \Psi_L$. Also as it has also Z and not $\bar{Z}$ﬄ it cannot interchange flavor with itself (you need a Z and $\bar{Z}$ for this), $K_{l,l+1}\Psi_L = 0$. With these properties and acting with the anomalous dimension operator on a chiral primary, we get:

$$\Gamma\Psi_L = \frac{\lambda}{16\pi^2}\sum_{l=1}^{L}\left(2-1+C\right)\Psi_L \tag{3.19}$$

Last step is to use the fact that the chiral primaries have no anomalous dimension. So we understand that we need to have $C=-1$.

We have determined the anomalous dimension. Let's see the result more abstractly. In general, we can map the operators we studied, the single-trace operators, to a tensor product of different Hilbert spaces.

$$\mathcal{V}_1\otimes\ \mathcal{V}_2\otimes ...\otimes\mathcal{V}_L \tag{3.20}$$

Due to the cyclicity property of the trace, the theory is invariant under the shift:

$$\mathcal{V}_1\otimes\ \mathcal{V}_2\otimes ...\otimes\mathcal{V}_L \rightarrow \mathcal{V}_L\otimes\mathcal{V}_1\otimes ..\otimes\mathcal{V}_{L-1} \tag{3.21}$$

Another property is that the operator is Hermitian. These are some indications that we can map the anomalous dimension to the energy spectrum of a spin chain. We studied only the scalar field behaviour (we will explain shortly the sub-sector of the whole theory). But we can generalize and conjecture that the planar $\mathcal{N}$=4 SYM is mapped fully to a psu(2,2|4) spin chain:

This is the novel thing! We started working on $\mathcal{N}$=4 SYM and the final result ended up being a calculation on a spin chain. The importance of this result isn't just the elegance of mapping two completely different theories to each other and therefore that we can see something with a new perspective. It is an additional computational comfort. As we know, a spin chain system is integrable and therefore we can solve it analytically. If we see the result we aqcuired for the anomalous dimension operator, we understand that we need to solve this for any spin site, which from a first glance seems a (very very)

Table 1: Dictionary of the two theories

Planar $\mathcal{N}$=4 SYM	**psu(2,2|4) spin chain**
Single-trace operators	Cyclic spin chain
Field	Spin site
Anomalous dilatation operator	Hamiltonian
Anomalous dimension	Energy
Cyclic constraint	Zero-momentum condition

long calculation if we take a considerably large number of fields in the trace. But via integrability in a spin chain, we bypass this problem, as we can use the techniques that have been constructed and get much easier the final answer.

We considered our trace operators to be consisted only of scalar fields. The symmetry group of R-symmetry is SO(6). Therefore the spaces $\mathcal{V}_i$ will correspond to an SO(6) vector representation. Consequently, we have to do with a SO(6) spin chain. The way that can be solved this via integrability is the algebraic Bethe ansatz. Because the symmetry group is pretty large and this will lead to nested expressions, we opt for solving a lower part of the theory, the SU(2) spin chain, the Heisenberg chain. We consider this as a pretty instructive way to present the basic ideas of integrability, without being messed up with exhausting calculations.

4 Heisenberg spin chain

We mentioned that instead of SO(6) sector (the set of scalar fields), we are going to find the energy spectrum for the SU(2), a subsector. In particular, the fields that any operator has are $Z = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ and $W = \frac{1}{\sqrt{2}}(\phi_3 + i\phi_4)$. According to the mapping to the spin chain, we can claim that Z corresponds to spin up, while W to spin down. Moreover, as an operator can have only those two fields (and not even their conjugates), it's easy to understand that $K_{l,l+1} = 0$, as you cannot contract Z with Z or W. Therefore the anomalous dimension is:

$$\Gamma_{SU(2)} = \frac{\lambda}{8\pi^2}\sum_{l=1}^{L}(1 - P_{l,l+1}) \tag{4.1}$$

If we want to write this in terms of spin chain, the Heisenberg model is:

$$\Gamma_{SU(2)} = \frac{\lambda}{8\pi^2}\sum_{l=1}^{L}\left(\frac{1}{2} - 2\vec{S}_l \cdot \vec{S}_{l+1}\right) \tag{4.2}$$

Moreover, expanding to its components (instead of S_x etc we will write in terms of S_+ etc components):

$$\Gamma_{SU(2)} = \frac{\lambda}{4\pi^2}\sum_{l=1}^{L}\Big(\frac{1}{4} - \frac{1}{2}\big(S_l^+ S_{l+1}^- + S_l^- S_{l+1}^+\big) - S_l^z S_{l+1}^z\Big) \tag{4.3}$$

The commutation relations they obey:

$$[S_j^z, S_k^\pm] = \pm\hbar\delta_{j,k}S_j^\pm \quad [S_j^+, S_k^-] = 2\hbar\delta_{j,k}S_j^z \tag{4.4}$$

Going to construct the Hilbert space, we understand that each site has two states. We chose these states to be eigenstates of the S^z_j operator. So the basic properties of the Hilbert space of each site are:

$$\mathcal{H}_j : \quad S^z_j \left|\pm\right\rangle_j = \pm\frac{1}{2}\left|\pm\right\rangle_j, \quad S^{\pm}_j \left|\mp\right\rangle_j = \left|\pm\right\rangle_j, \quad S^{\pm}_j \left|\pm\right\rangle_j = 0 \tag{4.5}$$

To obatin the full Hilbert space, $\mathcal{H} = \oplus_{j=1}^{L}\mathcal{H}_j$. So a state of the spain is the tensor product of states in each site. Next step is to define the vacuum. As we have a ferromagnet, we define the ground state as the state with spins pointing up along z-axis.

$$\left|0\right\rangle = \otimes_{j=1}^{L}\left|+\right\rangle_j \tag{4.6}$$

Also as we have $[H, S^z_{tot}] = 0$, where $S^z_{tot} = \sum_{j=1}^{L} S^z_j$. So every energy eigenstate has also fixed magnetization along the z-axis. So it is convenient to label energy states of the chain with the total magnetization. In particular, we can refer to a subspace as $\mathcal{H}_M$, where M indicates the number of down spins. So in order to study the Heisenberg chain, we have to take each sector $\mathcal{H}_M$. The first trivial case is to study the case where every spin site is aligned to z axis. It's obvious that the total spin will be $S^z_{tot} = \frac{L}{2}$ and the energy will be E=0. Next step is to study when there is one anti-aligned spin (M=1). Initially, for the eigenstate we will use the ansatz:

$$\left|\Psi_1\right\rangle = \sum_{j=1}^{L}\Psi_1(j)\left|j\right\rangle, \quad \left|j\right\rangle = S^-_j\left|0\right\rangle \tag{4.7}$$

What we did is just to write a general linear combination of states of the chain where there is only one spin down. If we take the Schrodinger equation $H\left|\Psi_1\right\rangle = E_1\left|\Psi_1\right\rangle$ and project it onto $\left\langle j\right|$:

$$\frac{\lambda}{8\pi^2}\big(\Psi_1(j-1) + \Psi_1(j+1)\big) = (-E_1 + \frac{\lambda}{4\pi^2})\Psi_1(j) \tag{4.8}$$

So the next step is to consider some function Ψ_1. A reasonable choice is the free wave ansatz $\Psi_1(j) = e^{ikj}$. With this, the eigenvalue will be:

$$E_1 = \frac{\lambda}{2\pi^2}sin^2(\frac{k}{2}) \tag{4.9}$$

Because of the cyclicity of the spin chain, we have the periodicity condition, which arises from the coupling of Lth site with the 1st. These conditions are:

$$\frac{\lambda}{8\pi^2}\big(\Psi_1(L) + \Psi_1(2)\big) = (-E_1 + \frac{\lambda}{4\pi^2})\Psi_1(1) \tag{4.10}$$

$$\frac{\lambda}{8\pi^2}\big(\Psi_1(1) + \Psi_1(L-1)\big) = (-E_1 + \frac{\lambda}{4\pi^2})\Psi_1(L) \tag{4.11}$$

This condition is equivalent with imposing the requirement $\Psi_1(j+L) = \Psi_1(j)$. So we have an additional condition for the momentum, which is quantized due to this.

Specifically, we have $e^{ikL} = 1$, which gives:

$$k = \frac{2\pi n}{L}, \quad n\epsilon[0, ..., L-1] \tag{4.12}$$

But we have to pay attention. We mapped the single-trace operators to spin sites, but moreover the dictionary has the condition that the momentum of the spin chain has to be zero (See the table in the previous section). So from a first glance, someone would say that the only acceptable mode is for n=0. But this case is trivial as every operator that obeys this condition is chiral primary and therefore its anomalous dimension will be zero. It is easy to see this from the result of the energy eigenvalue and due to the mapping we did between the energy eigenvalue and the anomalous dimension.

We continue for M=2. Based on this, we will generalize the result to arbitrary number of down spins (W fields). Similarly, the eigenstate will have the form $|\Psi_1\rangle = \sum_{j_1<j_2} \Psi_2(j_1, j_2)\, |j_1, j_2\rangle$ with $|j_1, j_2\rangle = S^-_{j_1} S^-_{j_2} |0\rangle$. The only thing we consider is for $j_1 = j_2$, then $\Psi_2 = 0$. So this way of writing is pretty general and doesn't make any prior assumption and doesn't restrict the validity for the whole problem. Again taking Schrodinger equation and projecting onto $\langle j_1, j_2|$:

$$\begin{aligned} \frac{\lambda}{8\pi^2}\big(\Psi_2(j_1-1, j_2) + \Psi_2(j_1+1, j_2) + \Psi_2(j_1, j_2-1) + \Psi_2(j_1, j_2+1)\big) = \\ \big(-E_2 + \frac{\lambda}{2\pi^2}\big)\Psi_2(j_1.j_2) \quad 2 < j_1+1 < j_2 < L \\ \frac{\lambda}{8\pi^2}\big(\Psi_2(j_1, j_2+1) + \Psi_2(j_1-1, j_2)\big) = (-E_2 + \frac{\lambda}{4\pi^2})\Psi_2(j_1, j_2) \\ 2 < j_1+1 = j_2 < L \end{aligned}$$

The corresponding free wave ansatz for Ψ_2 is:

$$\Psi_2(j_1, j_2) = A_{12} e^{ik_1 j_1 + ik_2 j_2} + A_{21} e^{ik_2 j_1 + ik_1 j_2} \tag{4.13}$$

The eigenvalues of energy are:

$$E_2 = \frac{\lambda}{2\pi^2}\Big(sin^2(\frac{k_1}{2}) + sin^2(\frac{k_2}{2})\Big) \tag{4.14}$$

The form of Ψ_2 doesn't depend on where the two down spins are. Thus if we use the second equation (using that $j_1 + 1 = j_2$) that we wrote previously, we can get some relation between A_{12} and A_{21}.

$$\frac{A_{12}}{A_{21}} = -\frac{1 + e^{i(k_1+k_2)} - 2\Delta e^{ik_1}}{1 + e^{i(k_1+k_2)} - 2\Delta e^{ik_2}} \equiv -e^{-i\phi(k_1,k_2)} \tag{4.15}$$

We have defined the quantity $\phi(k_1, k_2)$:

$$\phi(k_1, k_2) \equiv 2arctan\left(\frac{\Delta sin\frac{k1-k2}{2}}{cos\frac{k_1+k_2}{2} - \Delta cos\frac{k_1-k_2}{2}}\right) \tag{4.16}$$

Having this, we can write the ansatz wavefunction-up to a global phase:

$$\Psi_2(j_1, j_2) = e^{i(k_1 j_1 + k_2 j_2 - \frac{\phi(k_1,k_2)}{2}} - e^{i(k_1 j_2 + k_2 j_1 + \frac{\phi(k_1,k_2)}{2})} \tag{4.17}$$

In order to solve fully the problem, we need additionally to take into cosideration the boundary conditions, that is $\Psi_2(j_2, j_1 + L) = \Psi_2(j_1, j_2)$ and $\Psi_2(j_2 - L, j_1) = \Psi_2(j_1, j_2)$. If we traslate these conditions in terms of momenta, we get the Bethe equations:

$$Lk_1 + \phi(k_1, k_2) = 2\pi(n_1 + \frac{1}{2}) \quad Lk_2 - \phi(k_1, k_2) = 2\pi(n_2 + \frac{1}{2}) \quad n_1, n_1 \epsilon \mathbb{Z} \tag{4.18}$$

Again we have to use the zero momentum condition. This will mean $k_1 = -k_2$. This means $\phi(k_1, k_2) = -k_1$. Moreover, the possible values for momenta are $\frac{2\pi(n+1/2)}{L-1}, n\epsilon\mathbb{Z}$. So this mean that the energy eigenvalues (anomalous dimension) we found previously become:

$$\gamma = E_2 = \frac{\lambda}{\pi^2} sin^2 \frac{\pi n}{L-1} \tag{4.19}$$

We studied the M=2 sector. The generalization from now on is much easier. Following the same train of thought, for some number of down spins M (M≤N/2) the ansatz wavefunctions are written:

$$\Psi_M(j_1, .., j)_M = \prod_{M \geq a \geq b \geq 1} sgn(j_a - j_b) \times$$
$$\sum_{P_M} (-1)^{[P]} e^{i \sum_{a=1}^{M} k_{P_a} j_a + \frac{i}{2} \sum_{M \geq a \geq b \geq 1} sgn(j_a - j_b) \phi(k_{P_a}, k_{P_b})}$$

The coupled Bethe equations then are given:

$$Lk_a + \sum_b \phi(k_a, k_b) = 2\pi I_a \tag{4.20}$$

Note that I_a is some half-integer number if M is even and integer if M is odd. The energy eigenvalues are:

$$E_M = \frac{\lambda}{2\pi^2} \sum_{a=1}^{M} sin^2 \frac{k_a}{2} \tag{4.21}$$

The total momentum is the sum of all k_i, but we impose this to be zero.

One last thing, it is commonly used in this kind of problems to work with rapidities. These are defined in the following way:

$$e^{ip} = \frac{u + \frac{i}{2}}{u - \frac{i}{2}} \Leftrightarrow u \equiv \frac{1}{2} cot \frac{k}{2} \tag{4.22}$$

If we want to write the Bethe equations in terms of rapidities:

$$\left(\frac{u_a + \frac{i}{2}}{u_a - \frac{i}{2}} \right)^L = \prod_{b \neq a}^{M} \quad a = 1, .., M \frac{u_a - u_b + i}{u_a - u_b - i} \tag{4.23}$$

Additionally, the energy/anomalous dimension will be given:

$$\gamma = E_M = \frac{\lambda}{8\pi^2} \sum_{j=1}^{M} \frac{1}{u_j^2 + \frac{1}{4}} \tag{4.24}$$

If we take the boundary conditions, the momentum will be given by:

$$P = \sum_{a=1}^{M} \frac{1}{i} ln\Big(\frac{u_a + \frac{i}{2}}{u_a - \frac{i}{2}}\Big) = \pi M - \frac{2\pi}{N} \sum_{a=1}^{M} I_a \quad mod 2\pi \tag{4.25}$$

The number I_a is some half-odd integer for L-M even and integer for L-M odd. But due to the condition for zero momentum, this condition turns out to be:

$$\prod_{a=1}^{M} \Big(\frac{u_a + \frac{i}{2}}{u_a - \frac{i}{2}}\Big) = 1 \tag{4.26}$$

5 Conclusion

We studied the $\mathcal{N} = 4$ SYM and found that we can compute quantities with the help of integrability. Specifically, we mapped the problem of computing one-loop anomalous dimensions of single trace scalar operators to the computation of the energies of a spin chain. In this paper we only dealt with the $SU(2)$ sector, since in this case we have the Heisenberg spin chain, and hence it is the best choice for illustrating how integrability enters in $\mathcal{N} = 4$ SYM. However, it is possible to repeat the same process for the $SO(6)$ sector [10]. It is also possible to go beyond the first loop. In [6] the two and three-loop anomalous dimension was calculated and it was again proved to be integrable. Moreover, in [9] the anomalous dimension for the whole $PSU(2, 2|4)$ of the $\mathcal{N} = 4$ SYM was calculated. Taking all these into consideration, we can conjecture that integrability remains in all loops.

Our paper focused on the one side of the field called AdS/CFT integrability. The AdS/CFT comes into play, because the conformal dimensions of the gauge invariant operators of the $\mathcal{N} = 4$ SYM are associated with the energies of string states in the AdS side. It is also believed that other examples of the AdS/CFT correspondence are integrable in the large N limit as well. All these help us realize the importance of integrability in the high energy physics domain, since it helps us compute quantities in regions otherwise difficult to approach.

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