Student Seminar
Entanglement Entropy and CFT

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1 Introduction

The aim of this digest is to summarize in an understandable way, to the extent that this is possible, our first conclusions stemming from literature research on the topic of entanglement entropy and CFT. In this context we have decided that the optimal approach is to attempt to convey the limited knowledge we have gained in a simple and intuitive manner and risk losing some of the mathematical rigor that is in many cases essential to this particular subject. Of course that decision relies on the fact that this work is intended for our fellow students and therefore it merely has to act as a guide into the (in part) uncharted territory of our topic. Of course all the information that we will not include explicitly can be found in the provided references, where the interested reader will have the chance to form a more complete picture and recover the level of detail that this work might be lacking. Nevertheless, the calculations that are present in the following sections have been treated with utmost care so as to be clear, detailed and in the cases that this is feasible self-contained in order to avoid overloading the reader with ex-machina type of justifications for the steps that we follow. Still, this part of Physics is related to many abstract notions which are not yet fully understood by the scientific community. Entanglement entropy itself is one of these concepts and precisely because it is an object of active research it is worth the time and the effort to at least understand the very basics about it.

Both entanglement and entropy are two notions that have played an important role in Physics since they are essential for understanding the microscopic details of the universe. Within the context of our topic their union in the form of entanglement entropy has proven to be an important tool when it comes to questions about the information contained in the systems that we will be considering. It is undeniable that problems related to information have dominated the last decades of research either in the form of fundamental questions, like understanding black holes and the information paradox, or in the form of more concrete matters, for example how we can store and handle information when it comes to quantum computing applications. Entanglement entropy lies in the center of many such problems and therefore it is crucial for anyone who aspires to become involved with them to familiarize themselves with it. Furthermore, it is present both in high and low energy Physics and thus it can serve as a link between the two by providing either an intuitive way of thinking or in some cases a direct passage from the one to the other.

For all these reasons (but mainly because it's cool) we believe that it is indeed worth exploring this topic. Our approach will be to initially present the basic concepts regarding entanglement entropy and subsequently discuss its place in the CFT framework. Finally, some modern ideas about the relation between entanglement entropy and the AdS/CFT conjecture will be presented along with possible future developments in this field.

2 Entanglement Entropy

2.1 Key notions

Before actually treating the problem at hand it would be useful to present some of the concepts that will be needed along the way. The first such quantity is the density matrix which is a tool for the description of mixed states (that is statistical ensembles over quantum states) and is defined as:

$$\rho = \sum_i p_i \, |\psi_i\rangle\langle\psi_i| \tag{1}$$

Where each ψ_i represents a pure state and p_i the corresponding probability associated with its appearance in the ensemble. In order to avoid any possible confusion it should be emphasized that a mixed state, described by a density matrix, and a superposition state of the form $|\psi\rangle = \sum_i c_i |\psi_i\rangle$ are fundamentally different and to illustrate this, one could consider the average value of an observable in each case. For the mixed state the average value of the observable A is going to be:

$$\langle A \rangle = \sum_i p_i \langle \psi_i | A | \psi_i \rangle \tag{2}$$

Which is just the sum of the average value of A over all the states i weighed by their respective probability. In contrast to the above, the corresponding result for the superposition state is

$$\langle A \rangle = \sum_i |c_i|^2 \langle \psi_i | A | \psi_i \rangle + \sum_{i \neq j} c_i^* c_j \langle \psi_i | A | \psi_j \rangle \tag{3}$$

The first part is the same as before since $|c_i|^2$ is just the probability p_i, but the second contains what is referred to as interference terms which come from the non diagonal terms of the operator A. Thus, there is a manifest difference between the cases where we consider an ensemble and a superposition of states, which can be neatly summarized in the following statement: The ensemble describes N copies of a quantum system, whereas a superposition describes a single quantum system that consists of M interfering pure states.

The most important quantity that will be used throughout this digest is the Von Neumann entropy (or simply entanglement entropy), which in the framework of non-relativistic quantum mechanics is defined in terms of the density matrix as:

$$S = \mathrm{Tr}\, \rho \log \rho \tag{4}$$

2.1.1 Entanglement in quantum mechanics

Given two Hilbert spaces H_A, H_B, the most general state in their composite system $H_A \otimes H_B$ is:

$$|\psi\rangle_{AB} = \sum_{i,j} c_{ij} |A_i\rangle \otimes |B_j\rangle \tag{5}$$

Generally, the two subsystems are entangled, meaning that not all of the states can be written as product states $|A\rangle_i \otimes |B\rangle_j$ or equivalently that not all the coefficients c_{ij} can assume the form $c_{ij} = c_i^A c_j^B$. We can associate a density matrix to this pure state:

$$\rho = |\psi\rangle\langle\psi| \tag{6}$$

The density matrices satisfy the unitarity condition $\mathrm{Tr}\,\rho = 1$, so even though not all states can be separated one can focus on the description of one of the two subsystems, e.g. A, by tracing over the states of subsystem B. This procedure results in a new quantity called the reduced density matrix, which is defined as:

$$\rho_A = \mathrm{Tr}_B\,\rho \equiv \sum_k \langle B_k|\rho|B_k\rangle \tag{7}$$

Creating a reduced density matrix however, requires a "sacrifice" that might not be apparent from the above formalization. Namely, one has to omit any information contained in subsystem B, since tracing over its states is equivalent to the statement that B is inaccessible; in quantum mechanical terms this means that performing measurements in this part of the Hilbert space is in some sense forbidden. This is (or should be) surprising for two reasons:

- The splitting of the Hilbert space is not a physical process. In fact it can be a completely imaginary process [3] and this provides an intuitive explanation as to why entanglement entropy is a good measure of information: the larger the area of Hilbert space that cannot be probed the more information is withheld from the observer.
- There are certain instances when a natural boundary arises that obstructs us from gathering information from a specific region of spacetime. A prime example is (what else?) the event horizon of black holes. Thus, entanglement entropy qualifies as a candidate for the tools that one might use to study these objects.

As a remark, which we will use later, it is also important to notice that the tracing over B is equivalent (due to orthonormalization of states) to "identifying" the B's inside the full density matrix. If we apply this to the original state (5), it yields:

$$\rho_A = \sum_{i,j} d_{ik}\,|A_i\rangle\,\langle A_k|\,, \quad d_{ik} = \sum_l c_{il}c_{kl}^* \tag{8}$$

2.1.2 Entropy as a measure of entanglement

The entropy is a magnitude related to the amount of information in a system, and therefore it is perfectly suited to study entangled states. The entropy of one subsystem will determine its degree of entanglement with the other.

We will be using the von Neumann entropy :

$$S_A = -\operatorname{Tr}_A \rho_A \log \rho_A \tag{9}$$

(With the concept of the logarithm of a matrix understood as the logarithm of its eigenvalues)

The entanglement entropy, at zero temperature, also has an important property [2]: If B is the complement of A, then $S_A = S_B$. This will serve as a check for our results.

If we write $\operatorname{Tr} \rho_A^n$, with $n \in \mathbb{N}$, we know that the sum (trace) will converge, as the eigenvalues of the density matrix $\lambda_i \in [0,1]$, and therefore $\sum_i \lambda^n$ will converge. Now, we can analytically extend n to be real, take a derivative with respect to it (as the derivate is linear, it commutes with the trace), and then set $n = 1$, and the result is (up to a sign) the von Neumann entropy:

$$S_A = -\frac{\partial}{\partial n} \operatorname{Tr}_A \rho_A^n \bigg|_{n=1} = -\operatorname{Tr}_A \rho_A^n \log \rho_A \bigg|_{n=1} = -\operatorname{Tr}_A \rho_A \log \rho_A$$

Now we want to generalize these results to the quantum field theory formalism, so that we can use all the CFT tools.

2.2 QFT description and the replica trick

Our goal will be computing the previously mentioned $\operatorname{Tr}_A \rho_A^n$. We start defining the full density matrix with the (euclidean) path integral formalism:

$$\rho(\{\phi_x\}|\{\phi'_{x'}\}) = \frac{1}{Z} \int [d\phi(y,\tau)] \prod_{x'} \delta(\phi(y,0) - \phi'_{x'}) \prod_{x} \delta(\phi(y,\beta) - \phi_x) e^{-S_E} \tag{10}$$

This definition indeed satisfies the unitarity property of the density matrix $\mathrm{Tr}\,\rho = 1$, which in this language is equivalent to identifying 0 with β, leading to gluing ϕ_x to $\phi'_{x'}$, thus obtaining the partition function Z as the result of the integral, and therefore the tracing yields $Z/Z = 1$.

The next step is straightforward, we need the reduced density matrix, so we trace over B. This is the point where the equivalent interpretation of this tracing comes into play: We identify (or sew together, using terminology from the literature) the points x that are not in A, which pictorially leads to an image of a cylinder with open cuts at imaginary time $\tau = 0$ (this choice is arbitrary) for the intervals corresponding to A, at which the fields are not identified. We have just obtained the reduced density matrix ρ_A.

Pictorially, we are performing the operation:

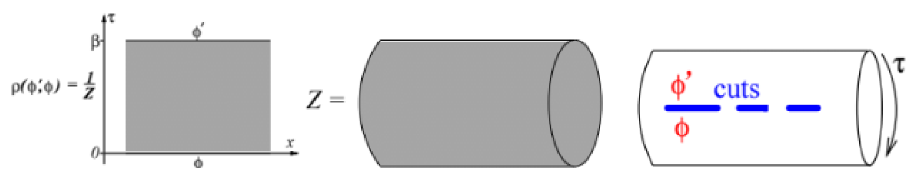

Figure 1: The leftmost picture is the representation of the density matrix. The other two represent the tracing over B, leaving the cuts over A [1]

Now we need the trace of n of these cylinders: this is equivalent to identifying the field in one cylinder to the field in the next one, and so on, until the last one is glued back to the first one:

$$\phi_i(x, 0^+) = \phi_{i+1}(x, 0^-), \quad x \in A, \quad i = 1, ..., n; \quad i + n \equiv i \tag{11}$$

where the second argument is the τ-coordinate of the field, meaning that the field from one cylinder is glued to the field from the next one, both at the same $\tau = 0$ but from a different sides.

We can understand now that $\mathrm{Tr}_A\,\rho_A^n = \dfrac{Z_n(A)}{Z^n}$, where Z is the usual partition function, and Z_n is the partition function of this n-cylinder-connected surface needed in the replica trick.

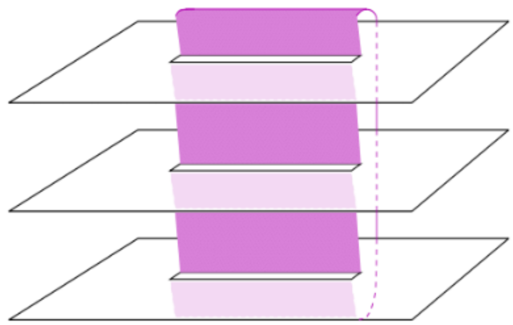

Figure 2: Pictorial representation of the n-sheeted surface $\mathcal{R}_n$ of the replica trick, for n=3 cylinders, again found in [1]

Notice that A might be a collection of intervals (as shown in Figure 1), but in our case we will just study the 1-interval ($x \in [u, v]$) case, as it is already challenging enough, although the extension to the N-interval case can be found in the references provided [1, 2].

2.3 CFT approach

From now on we will be using the machinery coming from Conformal Field Theories (in 2 dimensions, for simplicity). But first, we need to change the problem from the complicated topology of this n-sheeted surface, to the target space of our field theory, and in order to achieve that we need some auxiliary fields to take account of this topology: these are the so-called *twist fields*, $\mathcal{T}_n, \mathcal{T}_{-n}$, which are inserted on each sheet (treated as a complex plane) and substitute the conditions (11).

It can be proven [1] that this trace can be expressed as proportional to the product of the correlation function of these fields for each sheet, namely:

$$\mathrm{Tr}_A\, \rho_A^n \propto \prod_{i=1}^{n} \langle \mathcal{T}_n^i(u) \mathcal{T}_{-n}^i(v) \rangle = \langle \mathcal{T}_n(u) \mathcal{T}_{-n}(v) \rangle^n \tag{12}$$

And also, they allow us to express any expectation values on the n-cylinder in terms of the expectation values on the complex plane:

$$\langle O(w)\rangle_{\mathcal{R}_n} = \frac{\langle O(w)\mathcal{T}_n(u)\mathcal{T}_{-n}(v)\rangle_{\mathbb{C}}}{\langle \mathcal{T}_n(u)\mathcal{T}_{-n}(v)\rangle_{\mathbb{C}}} \tag{13}$$

Now our only task is to compute the two point function of these twist fields. It is clear now why we ask for the aid of CFT: computing two point functions of primary operators is rather straightforward. In this case:

$$\langle \mathcal{T}_n(u)\mathcal{T}_{-n}(v)\rangle_{\mathbb{C}} = \frac{1}{(u-v)^{2\Delta_n}(\bar{u}-\bar{v})^{2\bar{\Delta}_n}} \tag{14}$$

2.3.1 Weight of the twist operators

The question that remains is: which is the weight of these twist operators? To answer this question, we are going to use our friend, the energy-momentum tensor $T(z)$. First of all, the expectation value of this operator in the complex plane $\mathbb{C}$ is zero $\langle T(z)\rangle_{\mathbb{C}} = 0$. Now we perform a conformal map from the n-sheeted surface $\mathcal{R}_n$, to the plane. As we know, the energy-momentum tensor has the following transformation law under conformal transformations:

$$w \to z(w): \quad T(w) \to \left(\frac{\partial z}{\partial w}\right)^2 T(z) + \frac{c}{12}S[z,w] \tag{15}$$

where $S[z,w]$ is the *Schwartzian derivative*, defined as:

$$S[z,w] \equiv \frac{\partial_w^3 z}{\partial_w z} - \frac{3}{2}\left(\frac{\partial_w^2 z}{\partial_w z}\right)^2 \tag{16}$$

The mapping we are interested in is the mapping from $\mathcal{R}_n$ to the complex plane, given by:

$$z(w) = \left(\frac{w-u}{w-v}\right)^{1/n} \tag{17}$$

After a tedious calculation (which can be found in B.1), the corresponding Schwartzian derivative is:

$$S[z,w] = \frac{1}{2}\left(1-\frac{1}{n^2}\right)\frac{(u-v)^2}{(w-u)^2(w-v)^2} \tag{18}$$

If we now gather this along with the vanishing expectation value of T in the complex plane, it's expectation value in the n-cylinder:

$$\langle T(w)\rangle_{\mathcal{R}_n} = \cancel{\left(\frac{\partial z}{\partial w}\right)^2 \langle T(z)\rangle} + \left\langle \frac{c}{12} S[z,w] \right\rangle = \frac{c}{24}\left(1-\frac{1}{n^2}\right)\frac{(u-v)^2}{(w-u)^2(w-v)^2} \tag{19}$$

For reasons that will become clear later, we will define

$$\Delta_n = \bar{\Delta}_n \equiv \frac{c}{24}\left(1-\frac{1}{n^2}\right) \tag{20}$$

(spoiler alert: yes, the choice of Δ_n is not random, these will become the weight of the twist operators).

We are now pursuing the obtaining of the RHS of (13), as we already achieved the LHS. For that purpose, we now introduce the Conformal Ward Identity:

$$\langle T(z)\phi_1(z_1)\phi_2(z_2)\rangle = \sum_j \left(\frac{\Delta_j}{(z-z_j)^2} + \frac{1}{z-z_j}\partial_{z_j}\right)\langle \phi_1(z_1)\phi_2(z_2)\rangle \tag{21}$$

In our case ($z_1 \equiv u, z_2 \equiv v, \Delta_n = \Delta_{-n}$):

$$\langle T(w)\mathcal{T}_n(u)\mathcal{T}_{-n}(v)\rangle_{\mathbb{C}} = \sum_{j=1}^{2}\left(\frac{\Delta_j}{(w-z_j)^2} + \frac{1}{w-z_j}\partial_{z_j}\right)\langle \mathcal{T}_n(u)\mathcal{T}_{-n}(v)\rangle_{\mathbb{C}} \tag{22}$$

$$= \left(\frac{\Delta_n}{(w-u)^2} + \frac{1}{w-u}\partial_u + \frac{\Delta_{-n}}{(w-v)^2} + \frac{1}{w-v}\partial_v\right)\langle \mathcal{T}_n(u)\mathcal{T}_{-n}(v)\rangle_{\mathbb{C}} \tag{23}$$

Plugging in this expression the form of the correlator (14), and after another tedious and not insightful calculation (which can also be found in B.2), we obtain the following result:

$$\langle T(w)\mathcal{T}_n(u)\mathcal{T}_{-n}(v)\rangle_{\mathbb{C}} = \frac{\Delta_n}{(w-u)^2(w-v)^2(u-v)^{-2}}\langle \mathcal{T}_n(u)\mathcal{T}_{-n}(v)\rangle_{\mathbb{C}} \tag{24}$$

And with this we can now finally fulfill the promise of computing the RHS of equation (13):

$$\frac{\langle T(w)\mathcal{T}_n(u)\mathcal{T}_{-n}(v)\rangle_{\mathbb{C}}}{\langle \mathcal{T}_n(u)\mathcal{T}_{-n}(v)\rangle_{\mathbb{C}}} = \frac{\Delta_n}{(w-u)^2(w-v)^2(u-v)^{-2}} = \langle T(w)\rangle_{\mathcal{R}_n} \tag{25}$$

And, as we can see, our previous identification (20) of the weight of the twist operators is consistent with the expected result.

2.3.2 Entanglement entropy formula

Now that we have assured what we needed, we can just compute the trace of the n-sheeted density matrix:

$$\mathrm{Tr}_A\, \rho_A^n \propto \langle \mathcal{T}_n(u)\mathcal{T}_{-n}(v)\rangle^n = \left(\frac{1}{|u-v|^{4\Delta_n}}\right)^n = |u-v|^{-\frac{c}{6}\left(n-\frac{1}{n}\right)} \tag{26}$$

If we now define $l = |u-v|$ (the length of the interval), and set the proportionality constant to b_n, we can finally achieve our goal:

$$S_A = -\frac{\partial}{\partial n}\,\mathrm{Tr}_A\,\rho_A^n\Big|_{n=1} = -\frac{\partial}{\partial n}\left(b_n|u-v|^{-\frac{c}{6}\left(n-\frac{1}{n}\right)}\right)\Big|_{n=1} \tag{27}$$

$$= -\left[b_n\,\log(l)\,\frac{c}{6}\left(-1-\frac{1}{n^2}\right) l^{-\frac{c}{6}\left(n-\frac{1}{n}\right)} + a_n'\, l^{-\frac{c}{6}\left(n-\frac{1}{n}\right)}\right]\Big|_{n=1} \tag{28}$$

$$= b_1\frac{c}{3}\log(l) - b_1' \tag{29}$$

Few technical issues to address:

- We defined the two point function of the twist fields to be normalized (we set the constant numerator to 1), but if we want to be more precise, we can write an arbitrary constant a (an UV cutoff) instead.
- Unitarity ($\mathrm{Tr}\,\rho = 1$) implies that for $n = 1$, the proportionality constant b_1 has to be 1.

With this in mind, we now have as a final expression:

$$S_A = \frac{c}{3}\log\frac{l}{a} - b_1' \tag{30}$$

This formula raises some questions, which deserve some comments:

- The first term of the entropy has a very important and rather surprising feature: given a theory whose central charge c is known, it only depends on the length of the interval, no other parameters of the theory are involved. In other words, the operator content of the theory is not relevant. Moreover, it is evident that for $\frac{l}{\alpha} \Rightarrow \infty$ the entropy diverges; this generally does not pose a problem since there is a procedure for renormalizing the result, it is however indicative of the fact that the largest contribution to the entanglement entropy comes from the degrees of freedom that are close to the boundary separating the two subsystems. The second term on the other hand, is not as general, but it is known for some integrable models. In our case, it is not going to be relevant, so we are just going to drop it.

- The calculation we've gone through is the simplest case one can think of, with just one interval, at zero temperature, embedded in an infinite size 1D system $A \cup B$. Nonetheless, generalizations of this result to more complicated settings can be obtained with a bit of hard work.

  In particular, we will be interested in the next section in the finite size case. Again, CFT comes in handy as it provides a tool to easily obtain our result: the primary transformation law. Concretely, under a conformal map, the 2-point function of primary fields transforms as:

  $$\langle \mathcal{T}_n(z_1)\mathcal{T}_{-n}(z_2)\rangle = \left(\frac{\partial w_1}{\partial z_1}\right)^{2\Delta_n} \left(\frac{\partial w_2}{\partial z_2}\right)^{2\Delta_n} \langle \mathcal{T}_n(w_1)\mathcal{T}_{-n}(w_2)\rangle \tag{31}$$

  This allows us to easily compute the entropy for other geometries. If we choose $w \to z = (L/2\pi)\log w$, then the complex plane is mapped to a cylinder of circumference L, which is the compactification (finite size) we were interested in, and we get the entropy formula we were looking for:

  $$S_A = \frac{c}{3}\log\left(\frac{L}{\pi a}\sin\frac{\pi l}{L}\right) \tag{32}$$

  This formula has two interesting properties (that can also be seen as consistency checks), namely: 1) In the limit $l/L \ll 1$ it reduces to the infinite size formula (30); 2) The expression is invariant under $l \to L-l$, which is the previously mentioned requirement $S_A = S_B$.

Similar conclusions can be obtained for the non-zero temperature, bounded systems or disjoint intervals cases, which will not be discussed here, but can be found in [1, 2].

3 Entanglement Entropy in the AdS/CFT Framework

In the previous section we have shown how one can derive a formula for the entanglement entropy by working in the CFT framework. We will now exploit this result, along with the holographic dictionary, in order to understand what it implies for the dual AdS_3 space [3]. The interested reader can find more information on the matter by referring to [4].

3.1 The AdS Spacetime

Before addressing the task at hand it is required to provide some basic information on the AdS spacetime itself in order to familiarize the reader with some of the objects that will appear later on.

An AdS_{d+1} space is a maximally symmetric spacetime with negative cosmological constant, that can be embedded in a $d+2$ dimensional Minkowski spacetime $(X^0, X^1, ..., X^d, X^{d+1}) \in \mathbb{R}^{d,2}$ with metric $\bar{\eta} = diag(-,+,...,+,-)$ [4].The form of the metric is thus:

$$ds^2 = -(dX^0)^2 + (dX^1)^2 + ... + (dX^d)^2 - (dX^{d+1})^2 \equiv \bar{\eta}_{\mu\nu} dX^\mu dX^\nu \tag{33}$$

and the AdS space is given by the hypersurface:

$$\bar{\eta}_{\mu\nu} X^\mu X^\nu = -(X^0)^2 + \sum_{i=1}^{d} (X^i)^2 - (X^{d+1})^2 = R^2 \tag{34}$$

Where R is the radius of curvature of AdS. In Poincaré coordinates, the AdS metric can be written as:

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 - dt^2 + \sum_{i=1}^{d-1} dx_i^2\right) \tag{35}$$

In a different coordinate system (global coordinates) the metric for AdS takes the form:

$$ds^2 = R^2(-\cosh^2\rho dt^2 + d\rho^2 + \sinh\rho^2 d\Omega_d^2) \tag{36}$$

Unfortunately the only case of visualization that exists concerns AdS_2, and is given in figure 3 [4].

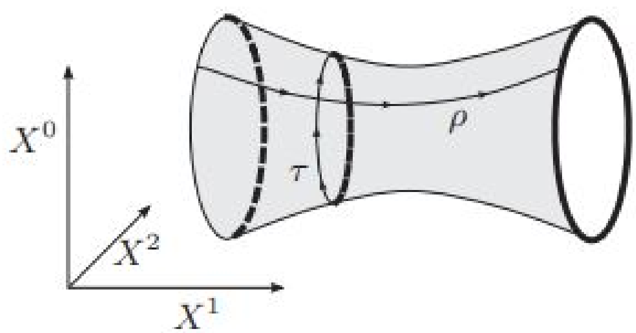

Figure 3: Pictorial representation of AdS_2 spacetime embedded in 1+2 dimensional Minkowski spacetime.

Even though we are not being very thorough, it is already evident from the figure that AdS space has rather bizarre geometrical properties that are periodic in time. Moreover, a feature that does not stem directly from the above, but it is worth mentioning nonetheless is the fact that light seems to be moving with superluminal velocities since it can travel to spatial infinity and back in a finite time interval [5]. However, despite its bizarre characteristics and the non-trivial geometry, one has to keep in mind that AdS is a solution to the vacuum Einstein equations and thus it requires modification in order to include objects of interest like black holes.

3.2 The Holographic Principle and ADS/CFT

The Holographic Principle was devised (mainly) by Gerardus 't Hooft and Leonard Susskind in the beginning of the 1990's, in the context of String Theory. It was motivated by the discovery, decades before, of the Bekenstein-Hawking black hole entropy, which takes the form:

$$S = \frac{A}{4G} \tag{37}$$

This is an upper bound to the entropy of a black hole, and was generalized by Bekenstein into what is known as the "Bekenstein bound", which states that the maximum entropy of a given system is determined by (37).

This lead to the proposal of the Holographic Principle, which is a property that has been conjectured to be fulfilled by any supergravity theory: all the information in a given volume V is encoded in its boundary $\partial V = A$ (this is the reason why it was called "holographic", as it resembles a hologram, which is a 2D surface encoding 3D information).

The first concrete realization of the Holographic Principle was given by Juan Maldacena in 1995, who proposed the well-known AdS/CFT correspondence. This duality means that physical quantities on a curved spacetime can be obtained through a quantum field theoretical calculation, and vice versa. More concretely, the spacetime has to be an Anti-de Sitter spacetime, and the quantum field theory living in its boundary has to have conformal symmetry.

This principle does not only have importance in the high energy (stringy) physics, as an important property of this correspondence is the fact that a strongly coupled (and therefore, hard) problem in the CFT side is weakly coupled in the volume AdS. For this reason, condensed matter problems involving strongly coupled CFT's could be solved via General Relativity techniques.

In order to achieve this equivalence, there's what is called the "holographic dictionary", which allows us to translate quantities between both sides, and from which we will use some of its "entries" in the following section.

3.3 Holographic entanglement entropy

Let us expand on the case of the CFT_2, at zero temperature with a single interval. We have already argued that the weight (or conformal dimension) of the twist field operators $\mathcal{T}_{\pm n}$ is given by equation (20). We have also shown that $\mathrm{tr}_A\, \rho_A^n$ is equivalent to the n products of the two point functions $\langle \mathcal{T}_n(u)\mathcal{T}_{-n}(v)\rangle^n$ which by using the AdS/CFT correspondence can be translated as follows [3]:

$$\langle \mathcal{T}_n(P)\mathcal{T}_{-n}(Q)\rangle^n \sim e^{-\frac{2n\Delta_n \cdot L_{PQ}}{R}} \tag{38}$$

Where L_{PQ} is the geodesic length between the points P and Q in the AdS space and R is the radius of curvature appearing in the expression for the metric (36).

Using the same procedure as in subsection 2.3.2, the entanglement entropy can be computed as:

$$S_A = -\frac{\partial}{\partial n}\, \mathrm{Tr}_A\, \rho_A^n \bigg|_{n=1} = -\frac{\partial}{\partial n} e^{-\frac{2n\Delta_n \cdot L_{PQ}}{R}} \bigg|_{n=1} \tag{39}$$

$$= 2\frac{\partial(n\Delta_n)}{\partial n}\frac{L_{PQ}}{R} e^{-\frac{2n\Delta_n \cdot L_{PQ}}{R}} \bigg|_{n=1} \tag{40}$$

$$= \left[\frac{c}{12}(1-n^{-2}) + \frac{c}{6}(n^{-2})\right]\frac{L_{PQ}}{R}\bigg|_{n=1} = \frac{c}{6}\frac{L_{PQ}}{R} \tag{41}$$

Once again using the AdS/CFT correspondence we have for the central charge:

$$c = \frac{3R}{2G_N^{(3)}} \tag{42}$$

And thus the entanglement entropy in AdS is just:

$$S_A = \frac{L_{PQ}}{4G_N^{(3)}} \tag{43}$$

This is by itself a result of great importance since it completes the establishment of a relation between the microscopic CFT theory and the theory that describes the macroscopic spacetime. Moreover, it is in many cases much easier to compute the geodesic length of a curve rather than a correlation function and therefore the above formula can prove to be an important tool when one wants to compute the entanglement entropy of a system.

An interesting question that cannot remain unanswered is whether this result can be generalized in higher dimensions. The answer is yes, even though we are not going to provide proof of it, the entanglement in a CFT_{d+1} assumes the form [3]:

$$S_A = \frac{Area(\gamma_A)}{4G_N^{(d+2)}} \tag{44}$$

3.3.1 Poincaré coordinates

Finally we need to show that when translated back in CFT, equation (43) yields the correct result derived in the previous section and given by (30). This is fairly simple; in Poincare coordinates the length of a geodesic line is given by (an explicit derivation can be found in appendix B.3):

$$\text{Length}(\gamma) = \int ds = 2R \log\left(\frac{l}{a}\right) \tag{45}$$

Therefore (43) becomes:

$$S_A = \frac{2R \log \frac{l}{\alpha}}{4G_N^{(3)}} = \frac{c}{3} \log \frac{l}{\alpha} \tag{46}$$

Where in the last step equation (42) has been used. Thus, the correct expression can be indeed recovered verifying the consistency of the conjecture relating the two theories.

3.3.2 Global coordinates

Again, and for completeness, we can obtain the entropy formula that was derived for the finite size system (32) via geodesic length, but for this we will need to use global coordinates of AdS (36). The reason why we choose this metric is that in its conformal boundary it is the same metric as the one from a cylinder (and thus a compactified system). We are not going to perform the calculation here, but for the interested reader it can be found in [2]:

$$S_A \approx \frac{2R}{4G_N^{(3)}} \log\left(\frac{L}{a}\sin\frac{\pi l}{L}\right) = \frac{c}{3}\log\left(\frac{L}{a}\sin\frac{\pi l}{L}\right) \tag{47}$$

4 Conclusion

Motivated by the above there are several interesting remarks that can be made regarding entanglement entropy and CFT. First and foremost we have shown that within the CFT framework, entanglement entropy depends exclusively on the subsystem length. The reason that this is so important is because entanglement entropy measures the degree of entanglement between two subsystems or in different words quantum correlations [6], thus providing a much "cleaner" way to get to the fundamental properties of the system compared to most other standard quantities, including correlation functions themselves.

Moreover, one of the features of entanglement entropy that renders it an interesting quantity in the AdS/CFT framework is that it scales like the area of the subsystem under consideration much like the Bekenstein-Hawking entropy. Establishing a firm connection between the two is harder than it seems for the reason that even though they look similar, they are not the same. In fact entanglement entropy can be obtained as the one-loop quantum correction to the BH entropy in the presence of matter fields [3]; this setting is evidently more complicated than the one considered above, since it entails significant changes in the CFT and its dual AdS space.

So far entanglement entropy has been treated in a purely theoretical framework, but it has to be noted that recent developments in the field of condensed matter have led to its direct measurement in quantum many-body systems [7]. That can serve to soothe our uneasiness regarding the possibility that such a quantity is a purely theoretical concept (or practically a figment of our imagination), with no hope of ever being measured and thus it also serves as a promising start to a novel way of approaching experimentally the problems of systems near criticality.

Appendices

A More about entropy

One might notice that the von Neumann entropy is just the quantum equivalent of the Shannon entropy [8] $H = \sum_i -p_i \log p_i$. In a classical context, entropy is closely related to the problem of arranging N particles in m states. The latter can have in general different energies and therefore different configurations of the particles will lead to a certain total system energy and therefore the realization of a specific macrostate. Entropy is defined as the natural logarithm of the number of different ways in which one can arrange the particles such that they lead to the same macrostate. The number of different configurations in the above scheme is given by:

$$\frac{N!}{N_1!N_2!...N_m!} \tag{48}$$

Therefore, by defining $p_i = \frac{N_i}{N}$, $E/N = \sum_i -p_i \log E_i$ and using the Stirling approximation for factorials [8] it is straightforward to obtain the Shannon entropy:

$$\frac{1}{N} \log \frac{N!}{N_1!N_2!...N_m!} = H + O(N^{-1} \log N) \tag{49}$$

Thus the origin of the von Neumann entropy becomes clear; the information about the possible states, their associated energy and probability is encoded in the density matrix operator and therefore the sum is substituted by a trace.

Another possible and useful definition of entropy can be extracted from the Rényi entropies of order α, defined as

$$S_\alpha = \frac{1}{1-\alpha} \log \mathrm{Tr}\, \rho_A^\alpha \tag{50}$$

with the advantage of avoiding the logarithm, and which is in fact equivalent to the von Neumann case in the limit $\alpha \to 1$.

B Explicit calculations

B.1 Schwartzian derivative

The Schwartzian derivative is defined as:

$$S[z,w] \equiv \frac{\partial_w^3 z}{\partial_w z} - \frac{3}{2}\left(\frac{\partial_w^2 z}{\partial_w z}\right)^2 \tag{51}$$

The mapping we are interested in is the mapping from the n-sheeted cylinder to the complex plane, given by:

$$z(w) = \left(\frac{w-u}{w-v}\right)^{1/n} \tag{52}$$

Let's compute the first three derivatives of this function:

$$\partial_w z = \frac{1}{n}\frac{u-v}{(w-v)^2}\left(\frac{w-u}{w-v}\right)^{\frac{1}{n}-1} \tag{53}$$

$$\partial_w^2 z = \frac{1}{n}\left(\frac{1}{n}-1\right)\frac{(u-v)^2}{(w-v)^4}\left(\frac{w-u}{w-v}\right)^{\frac{1}{n}-2} - \frac{2}{n}\frac{u-v}{(w-v)^3}\left(\frac{w-u}{w-v}\right)^{\frac{1}{n}-1} \tag{54}$$

$$\partial_w^3 z = \frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)\frac{(u-v)^3}{(w-v)^6}\left(\frac{w-u}{w-v}\right)^{\frac{1}{n}-3} \tag{55}$$

$$+ \frac{-6}{n}\left(\frac{1}{n}-1\right)\frac{(u-v)^2}{(w-v)^5}\left(\frac{w-u}{w-v}\right)^{\frac{1}{n}-2} \tag{56}$$

$$+ \frac{6}{n}\frac{u-v}{(w-v)^4}\left(\frac{w-u}{w-v}\right)^{\frac{1}{n}-1} \tag{57}$$

Now, the two different terms of the Schwartzian derivative:

$$\frac{\partial_w^3 z}{\partial_w z} = \left(\frac{1}{n} - 1\right)\left(\frac{1}{n} - 2\right)\frac{(u-v)^2}{(w-v)^2(w-u)^2} \tag{58}$$

$$+ (-6)\left(\frac{1}{n} - 1\right)\frac{u-v}{(w-v)^2(w-u)} + \frac{6}{(w-v)^2} \tag{59}$$

$$\frac{\partial_w^2 z}{\partial_w z} = \left(\frac{1}{n} - 1\right)\frac{u-v}{(w-v)(w-u)} - \frac{2}{w-v} \tag{60}$$

We need the square of this second term:

$$\left(\frac{\partial_w^2 z}{\partial_w z}\right)^2 = \left(\frac{1}{n} - 1\right)^2\frac{(u-v)^2}{(w-v)^2(w-u)^2} + \frac{4}{(w-v)^2} \tag{61}$$

$$+ (-4)\left(\frac{1}{n} - 1\right)\frac{u-v}{(w-v)^2(w-u)} \tag{62}$$

We can finally write the Schwartzian:

$$S[z,w] = \left(\frac{1}{n} - 1\right)\left(\frac{1}{n} - 2\right)\frac{(u-v)^2}{(w-v)^2(w-u)^2} \tag{63}$$

$$\cancel{+ (-6)\left(\frac{1}{n} - 1\right)\frac{u-v}{(w-v)^2(w-u)}} + \cancel{\frac{6}{(w-v)^2}} \tag{64}$$

$$- \frac{3}{2}\left(\frac{1}{n} - 1\right)^2\frac{(u-v)^2}{(w-v)^2(w-u)^2} - \cancel{\frac{3}{2}\frac{4}{(w-v)^2}} \tag{65}$$

$$\cancel{+ \frac{3}{2}4\left(\frac{1}{n} - 1\right)\frac{u-v}{(w-v)^2(w-u)}} \tag{66}$$

$$\tag{67}$$

Rearranging the non-vanishing terms:

$$S[z,w] = \left[\left(\frac{1}{n} - 1\right)\left(\frac{1}{n} - 2\right) - \frac{3}{2}\left(\frac{1}{n} - 1\right)^2\right]\frac{(u-v)^2}{(w-v)^2(w-u)^2} \tag{68}$$

$$= \frac{1}{2}\left(1 - \frac{1}{n^2}\right)\frac{(u-v)^2}{(w-v)^2(w-u)^2} \tag{69}$$

And we have therefore finished our task.

B.2 Conformal Ward Identity

$$\langle T(w)\mathcal{T}_n(u)\mathcal{T}_{-n}(v)\rangle_{\mathbb{C}} = \sum_{j=1}^{2}\left(\frac{\Delta_j}{(w-z_j)^2}+\frac{1}{w-z_j}\partial_{z_j}\right)\langle\mathcal{T}_n(u)\mathcal{T}_{-n}(v)\rangle \tag{70}$$

$$= \left(\frac{\Delta_n}{(w-u)^2}+\frac{1}{w-u}\partial_u+\frac{\Delta_{-n}}{(w-v)^2}+\frac{1}{w-v}\partial_v\right)\langle\mathcal{T}_n(u)\mathcal{T}_{-n}(v)\rangle \tag{71}$$

Plugging in this expression the form of the correlator

$$\langle\mathcal{T}_n(u)\mathcal{T}_{-n}(v)\rangle_{\mathbb{C}} = \frac{1}{(u-v)^{2\Delta_n}(\bar{u}-\bar{v})^{2\bar{\Delta}_n}} \tag{72}$$

and remembering that $\Delta_n = \Delta_{-n}$, we obtain the following:

$$\langle T(w)\mathcal{T}_n(u)\mathcal{T}_{-n}(v)\rangle_{\mathbb{C}} = \tag{73}$$

$$= \left(\frac{\Delta_n}{(w-u)^2}+\frac{-2\Delta_n}{(w-u)(u-v)}+\frac{\Delta_n}{(w-v)^2}+\frac{2\Delta_n}{(w-v)(u-v)}\right)\langle\mathcal{T}_n(u)\mathcal{T}_{-n}(v)\rangle_{\mathbb{C}} \tag{74}$$

If we already define $\langle T(w)\rangle_{\mathcal{R}_n} \equiv \langle T(w)\mathcal{T}_n(u)\mathcal{T}_{-n}(v)\rangle_{\mathbb{C}}/\langle\mathcal{T}_n(u)\mathcal{T}_{-n}(v)\rangle_{\mathbb{C}}$, operating with the previous expression leads to:

$$\frac{1}{\Delta_n}\langle T(w)\rangle_{\mathcal{R}_n} = \frac{1}{(w-u)^2}+\frac{-2}{(w-u)(u-v)}+\frac{1}{(w-v)^2}+\frac{2}{(w-v)(u-v)} \tag{75}$$

$$= \frac{(w-v)^2+(w-u)^2}{(w-u)^2(w-v)^2}+2\frac{(w-u)-(w-v)}{(w-u)(w-v)(u-v)} \tag{76}$$

$$= \frac{(w-v)^2+(w-u)^2-2(w-u)(w-v)}{(w-u)^2(w-v)^2} \tag{77}$$

$$= \frac{[(w-v)-(w-u)]^2}{(w-u)^2(w-v)^2} = \frac{(u-v)^2}{(w-u)^2(w-v)^2} \tag{78}$$

Summarizing, we found the following result:

$$\frac{\langle T(w)\mathcal{T}_n(u)\mathcal{T}_{-n}(v)\rangle_{\mathbb{C}}}{\langle\mathcal{T}_n(u)\mathcal{T}_{-n}(v)\rangle_{\mathbb{C}}} = \Delta_n\frac{(u-v)^2}{(w-u)^2(w-v)^2} \tag{79}$$

B.3 Length of geodesics (Poicaré coordinates)

In Poincaré coordinates (35) with t fixed, the geodesics are parametrized by:

$$(x, z) = \frac{l}{2}(\cos\lambda, \sin\lambda), \quad (\epsilon < \lambda < \pi - \epsilon) \tag{80}$$

where we have already introduced the cutoff $\epsilon \ll 1$, needed to regularize the metric (and therefore the quantities associated to it). We can obtain the line element ds^2 of the geodesic with this as:

$$ds^2 = \frac{R^2}{z^2}(dz^2 + dx^2), \quad (dx, dz) = \frac{l}{2}(-\sin\lambda\, d\lambda, \cos\lambda\, d\lambda) \tag{81}$$

Substituting both differentials:

$$ds^2 = \frac{R^2}{\sin^2\lambda}(\cos^2\lambda + \sin^2\lambda)d\lambda = \frac{R^2}{\sin^2\lambda}d\lambda \tag{82}$$

We can integrate now this, taking the cutoff into account:

$$\int ds = R\int_{\epsilon}^{\pi-\epsilon}\frac{d\lambda}{\sin\lambda} = 2R\int_{\epsilon}^{\pi/2}\frac{d\lambda}{\sin\lambda} \tag{83}$$

$$= -2R\log\left[\cot\lambda + \csc\lambda\right]\Big|_{\epsilon}^{\pi/2} \tag{84}$$

$$= -2R\left[\log(1) - \log(\cot\epsilon + \csc\epsilon)\right] \tag{85}$$

$$= 2R\log\left(\frac{1}{\epsilon} + \frac{1}{\epsilon} + O(\epsilon^2)\right) \approx 2R\log\left(\frac{2}{\epsilon}\right) \tag{86}$$

The cutoff in the boundary can be expressed as $\epsilon = 2a/l$, then:

$$\text{Length}(\gamma) = \int ds = 2R\log\left(\frac{l}{a}\right) \tag{87}$$

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