A CFT Approach to the Quantum Quench

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1 Introduction

Suppose we consider the time evolution of a d-dimensional quantum system from an initial state $|\psi_0\rangle$ which is the ground state of a hamiltonian H_0. At $t = 0$, one of the parameters of the hamiltonian is abruptly changed (for example we can inject energy or turn on a magnetic field). The abrupt change, or 'quench', is supposed to be carried out over a time scale much less than the dynamics near the ground state of H_0. How does the initial state $|\psi_0\rangle$ evolve under this new Hamiltonian $H \neq H_0$, i.e. what is $|\psi(t)\rangle = e^{-iHt}|\psi_0\rangle$?

This seems like a simple question that we have solved many times in our bachelor's quantum mechanics courses. However, in many-body systems, it has been shown to be a challenging problem to understand the dynamical behaviour following a quantum quench [4]. At the same time there are many reasons that make studying quantum quenches worthwhile. One of the scenarios in which the quantum quench gives us interesting physics is when we look for thermalisation.

If for an arbitrary finite subsystem A, the reduced density matrix $\rho_A(t)$ has a long-time limit that corresponds to the Gibbs statistical ensemble, the system is said to thermalise. In the recent theoretical and experimental research it has been shown that generic and integrable models have dramatically different behaviour following a quantum quench. Generic systems (locally) thermalise, whereas integrable models obtain stationary values that are described by an ensemble different from the Gibbs statistical ensemble [2]. This peculiar fact has revived the interest in the study of quantum quenches. Moreover, recent experimental developments have made it possible to create and study essentially one-dimensional systems by trapped ultra-cold atomic gases [2]. With this new experimental tool, it is possible to check and play with theoretical concept of quench dynamics. Hence, the quantum quench theory has possibly gained an important empirical foundation.

To study quench dynamics, advanced tools have been developed for analysing realistic models. However, as an introduction to the subject, it is wise to start with investigating quench dynamics in a 1+1D CFT. Not only is this setting a nice first step in the world of quantum quenches, it is also used as a playground for testing new ideas in quantum quenches. In this research digest, we would like to give an introduction of the 1+1D CFT approach to quantum quenches, following the paper by Pasquale Calabrese and John Cardy [1]. We want to make statements about the long-time evolution of correlation functions in a 1+1 dimensional CFT. Although this is of course a very specific model, we can make come some interesting observations on the behavior of these models after a quantum quench, such as the light-cone effect.

2 The general CFT approach to global quantum quenches

Firstly, we will outline the motivation for taking a CFT approach to studying the quantum quench. Suppose we consider a quantum theory living in d spatial dimensions, where time is taken to be continuous. We take a quantum system prepared at time $t = 0$ in a initial state $|\psi_0\rangle$, which is the ground state of some hamiltonian H_0, which we take to be translationally invariant, with short-range correlations and entanglement (e.g. the ground-state of a gapped hamiltonian H_0). We let this quantum state evolve (i.e. times $t > 0$) unitarily according to the dynamics given by a *different* Hamiltonian H, i.e. $H \neq H_0$. Though, this hamiltonian H may be related to H_0 by varying a parameter such as turning on an external field.

The correlation functions of some local operators $\Phi_j(\mathbf{r}_j)$ at some time t are the expectation values of the product of these operators given by [2]

$$\langle \Phi_n(t, \mathbf{r}_n) \rangle = \langle \psi_0 | \, e^{iHt} \Phi_1(\mathbf{r}_1) \Phi_2(\mathbf{r}_2) ... \Phi_n(\mathbf{r}_n) e^{-iHt} \, | \psi_0 \rangle \tag{1}$$

We can modify this expression to time-dependent correlation function in euclidean space (i.e. with imaginary time evolution) as

$$\langle \Phi_n(t, \mathbf{r}_n) \rangle = \langle \psi_0 | \, e^{-H\tau_2} \Phi_1(\mathbf{r}_1) \Phi_2(\mathbf{r}_2) ... \Phi_n(\mathbf{r}_n) e^{-H\tau_1} \, | \psi_0 \rangle \,, \tag{2}$$

where this equation is nothing more than the correlation function in an infinite euclidean strip of width $\tau_1 + \tau_2$ and with boundary conditions on each edge corresponding to the state $|\psi_0\rangle$. In order to return to the real time equation 1 from the equation for the strip geometry 2, we should analytically continue the imaginary times to real times. Naively, we would think the imaginary times transform as $\tau_1 \to it$ and $\tau_2 \to it$. However, doing so, we would end up with a euclidean strip of width zero, which doesn't make sense. Luckily, in the case when the time evolution is determined by a 1+1 dimensional CFT, we can avoid this problem. If a system is at or close to a quantum phase transition, we can appeal to the Renormalization Group (RG) theory of boundary critical phenomena: the actual boundary conditions $\tau = \tau_1$ and τ_2 are replaced by the conformal invariant boundary conditions $|\psi_0\rangle$ at $\tau = -\tau_0 + \tau_1$ and $\tau = \tau_0 + \tau_2$, where τ_0 is called the extrapolation length. We can make this replacement because we know that any boundary state will flow to one of the RG fixed points. These fixed points correspond to a conformal field theory and so we can use the powerful tools of CFT to evaluate our calculations. The extrapolation length τ_0 then measures the deviation of the actual state from the RG fixed points. Moreover, once we take a finite τ_0, it is then possible to have a nonzero width if we take $\tau_1 = it$ and $\tau_2 = -it$. It is not possible to take the limit $\tau_0 \to 0$ because scale-invariant boundary states are not normalizable and the subsequent time evolution would not be well defined.

To simplify the calculations we will perform later, it is practical to perform a translation in imaginary time $\tau \to \tau + (\tau_0 + \tau_1)$, which results in the following correlation function in the euclidean strip with $\tau \epsilon [0, 2\tau_0]$:

$$\langle\psi_0^*| \, \Phi_1(x_1,\tau)\Phi_2(x_2,\tau)...\Phi_n(x_n,\tau) \, |\psi_0^*\rangle \, ,$$

where the states $|\psi_0^*\rangle$ are the conformally invariant boundary states and τ must be considered a real number. Only at the end of the calculations we must analytically continue to real time by taking $\tau \to \tau_0 + it$ in order to evaluate the long (real) time behavior of the correlation functions.

3 One space dimension and CFT

We consider the case when H is at a quantum critical point whose long-distance behavior is given by a 1+1-dimensional CFT.

The infinite Euclidean strip can be formed from the upper half-plane (UHP) $\mathrm{Im} z > 0$ by the conformal mapping

$$w(z) = \frac{2\tau_0}{\pi} \log z, \tag{3}$$

where the points of the strip are labelled by the complex numbers $w_i = r_i + i\tau$ with $0 < \mathrm{Im}\, w < 2\tau_0$. On the UHP the images of the points at the same imaginary time on the strip lie along $\theta = \arg z_i = \pi\tau/2\tau_0$. This map is displayed in Figure 1.

In the case when our local operators are primary scalar operators $\Phi_i(w_i)$, the correlation function of these operators in the UHP is transformed to the correlation function in the strip by using the standard transformation

$$\langle \Pi \Phi_i(w_i)\rangle_{\mathrm{strip}} = \Pi |w'(z_i)|^{-x_i} \langle \Pi \Phi_i(z_i)\rangle_{\mathrm{UHP}}, \tag{4}$$

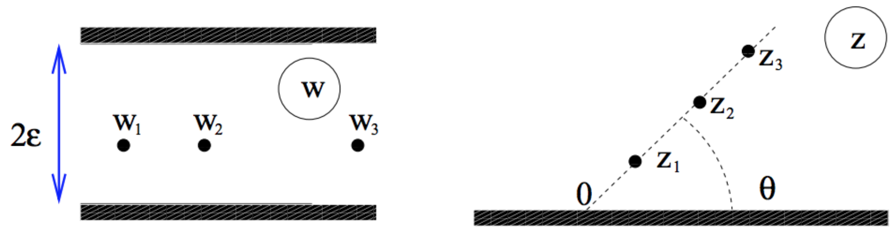

Figure 1: *Left:* Infinite imaginary time strip for points of the correlation functions at equal times, where $\tau = \operatorname{Im} w_i$ will be taken to real times, i.e. $\tau \to \tau_0 + it$. *Right:* Conformal mapping of the infinite time strip to the upper-half plane (c.f. Eq. 3), where $\theta = \pi\tau/2\tau_0$.

where the bulk scaling dimension of Φ_i is denoted by x_i. From Euclidean space we can go back to our Minkowski space by transforming $\tau \to \tau_0 + it$ and taking the limit $t, r_{ij} \gg \tau_0$.

This method is applied to the one-point and two-point function of a primary scalar field in the following sections.

3.1 The one-point function

We will start with considering the one-point function of a scalar primary field. In the UHP, the one-point function of a scalar primary field is

$$\langle \Phi(z) \rangle_{\mathrm{UHP}} = A_b^\Phi [2 \operatorname{Im} z]^{-x}, \tag{5}$$

where x denotes the bulk scaling dimension of the scalar primary field. The normalization factor A_b^Φ depends both on the field Φ and the boundary condition on the boundary b. When the primary field is not vanishing on the boundary, the conformal mapping from the UHP to the strip becomes

$$\begin{aligned}
\langle \Phi(z) \rangle_{\mathrm{strip}} &= \left| \frac{dw}{dz} \right|^{-x} \langle \Phi(z) \rangle_{\mathrm{UHP}} \\
&= A_b^\Phi \left| \frac{\pi z}{2\tau_0} \right|^{x} \left(2 e^{\frac{\pi r}{2\tau_0}} \sin \frac{\pi \tau}{2\tau_0} \right)^{-x} \\
&= A_b^\Phi \left(\frac{\pi}{4\tau_0} \right)^{x} \left| e^{\frac{\pi r}{2\tau_0}} e^{\frac{i\pi\tau}{2\tau_0}} \right|^{x} \left(e^{\frac{\pi r}{2\tau_0}} \sin \frac{\pi\tau}{2\tau_0} \right)^{-x} \\
&= A_b^\Phi \left[\frac{\pi}{4\tau_0} \frac{1}{\sin \frac{\pi\tau}{2\tau_0}} \right]^{x}.
\end{aligned}$$

Now we can continue to real time by plugging in $\tau = \tau_0 + it$. We use that

$$\sin \frac{\pi\tau}{2\tau_0} = \frac{1}{2i}\left(e^{\frac{i\pi\tau}{2\tau_0}} - e^{\frac{-i\pi\tau}{2\tau_0}}\right) \xrightarrow{\tau=\tau_0+it} \frac{1}{2i}\left(e^{\frac{i\pi\tau_0}{2\tau_0}} e^{\frac{-\pi t}{2\tau_0}} - e^{\frac{-i\pi\tau_0}{2\tau_0}} e^{\frac{\pi t}{2\tau_0}}\right) \tag{6}$$
$$= \frac{1}{2}\left(e^{\frac{-\pi t}{2\tau_0}} + e^{\frac{\pi t}{2\tau_0}}\right)$$
$$\approx \frac{1}{2} e^{\frac{\pi t}{2\tau_0}},$$

where in the last line we used that $t \gg \tau_0$. Hence, the one-point function becomes

$$\langle \Phi(t) \rangle \simeq A_b^{\Phi} \left(\frac{\pi}{2\tau_0}\right)^x e^{-x\pi t/2\tau_0}. \tag{7}$$

We see that observables described by a primary field, such as the order parameter, decay exponentially in time to zero (which is also the ground-state value). The relaxation time related to this process is $t_{\rm rel}^{\mathcal{O}} = 2\tau_0/x_{\mathcal{O}}\pi$ for an operator $\mathcal{O}$.

3.1.1 The energy density

The local energy density is an important exception to the exponential decay in time[2]. The energy density corresponds to the tt component of the energy-momentum tensor $T_{\mu\nu}$. In CFT this is not a primary operator. And consequently, if it is normalized so that $\langle T_{\mu\nu} \rangle_{\rm UHP} = 0$, in the strip we see that

$$\langle T_{tt}(\mathbf{r}, \tau) \rangle = \frac{\pi c}{24(2\tau_0)^2},$$

where c is the central charge of the CFT. As can be seen, this expectation value is not dependent on t, so it doesn't decay in time. Since the dynamics conserves energy, this is the result we would have expected. When taking the one-point functions of other globally conserverd quantities, which commute with the hamiltonian, similar features are expected.

3.2 The two-point function

3.2.1 The Gaussian Model

First, we consider the two-point function of a primary field in a boundary gaussian theory.

For a free boson the two point-function in the UHP is

$$\langle \Phi(z_1)\Phi(z_2)\rangle_{\text{UHP}} = \left(\frac{z_{1\bar{2}}z_{2\bar{1}}}{z_{12}z_{\bar{1}\bar{2}}z_{1\bar{1}}z_{2\bar{2}}}\right)^x, \tag{8}$$

where $z_{ij} = |z_i - z_j|$ and $z_{\bar{k}} = \bar{z}_k$. Also, note that Φ is not the gaussian field $\theta(z)$, but its exponential $\Phi(z) = e^{i\theta(z)}$ is.

The two-point function on the strip at equal imaginary time τ, at distance r apart under the conformal map 3 is given by

$$\begin{aligned}
\langle \Phi(r,\tau)\Phi(0,\tau)\rangle_{\text{strip}} &= \left|\frac{dw(z_1)}{dz_1}\right|^{-x}\left|\frac{dw(z_2)}{dz_2}\right|^{-x}\langle \Phi(z_1(w))\Phi(z_2(w))\rangle_{\text{UHP}} \\
&= \left|\frac{\pi z_1}{2\tau_0}\right|^{x}\left|\frac{\pi z_2}{2\tau_0}\right|^{x}\left(\frac{z_{1\bar{2}}z_{2\bar{1}}}{z_{12}z_{\bar{1}\bar{2}}z_{1\bar{1}}z_{2\bar{2}}}\right)^x \\
&= \left[\left(\frac{\pi}{2\tau_0}\right)^2 \frac{\cosh(\pi r/2\tau_0) - \cos(\pi\tau/\tau_0)}{8\sinh^2(\pi r/4\tau_0)\sin^2(\pi\tau/2\tau_0)}\right]^x,
\end{aligned} \tag{9}$$

where we have used that

$$\begin{aligned}
z_{1\bar{2}}z_{2\bar{1}} &= |z_1 - z_{\bar{2}}||z_2 - z_{\bar{1}}| \\
&= \left|e^{\pi r/2\tau_0}e^{i\pi\tau/2\tau_0} - e^{-i\pi\tau/2\tau_0}\right|\left|e^{i\pi\tau/2\tau_0} - e^{\pi r/2\tau_0}e^{-i\pi\tau/2\tau_0}\right| \\
&= 1 + e^{\pi r/\tau_0} - 2e^{\pi r/2\tau_0}\cos(\pi\tau/\tau_0) \\
&= e^{\pi r/2\tau_0}\left(e^{-\pi r/2\tau_0} + e^{\pi r/2\tau_0} - 2\cos(\pi\tau/\tau_0)\right) \\
&= 2e^{\pi r/2\tau_0}\left(\cosh(\pi r/2\tau_0) - \cos(\pi\tau/\tau_0)\right),
\end{aligned} \tag{10}$$

$$\begin{aligned}
z_{1\bar{1}}z_{2\bar{2}} &= |z_1 - z_{\bar{1}}||z_2 - z_{\bar{2}}| \\
&= e^{\pi r/2\tau_0}\left|e^{i\pi\tau/2\tau_0} - e^{-i\pi\tau/2\tau_0}\right|\left|e^{i\pi\tau/2\tau_0} - e^{-i\pi\tau/2\tau_0}\right| \\
&= 4e^{\pi r/2\tau_0}\sin^2(\pi\tau/2\tau_0)
\end{aligned} \tag{11}$$

and

$$
\begin{aligned}
z_{12}z_{\bar{1}\bar{2}} &= |z_1 - z_2||z_{\bar{1}} - z_{\bar{2}}| \\
&= \left|e^{\pi r/2\tau_0}e^{i\pi\tau/2\tau_0} - e^{i\pi\tau/2\tau_0}\right|\left|e^{\pi r/2\tau_0}e^{-i\pi\tau/2\tau_0} - e^{-i\pi\tau/2\tau_0}\right| \\
&= e^{\frac{\pi r}{\tau_0}} - 2e^{\frac{\pi r}{2\tau_0}} + 1 \\
&= 2e^{\frac{\pi r}{2\tau_0}}(\cosh\left(\frac{\pi r}{2\tau_0}\right) - 1) \\
&= 4e^{\frac{\pi r}{2\tau_0}} \sinh\left(\frac{\pi r}{4\tau_0}\right)^2,
\end{aligned}
\tag{12}
$$

where in the last line we have used $\cosh(2x) - 1 = 2\sinh(x)^2$.

Going to real time $\tau = \tau_0 + it$ yields for equation 9

$$
\langle\Phi(r,t)\Phi(0,t)\rangle_{\text{strip}} = \left[\left(\frac{\pi}{2\tau_0}\right)^2 \frac{\cosh(\pi r/2\tau_0) + \cosh(\pi t/\tau_0)}{8\sinh^2(\pi r/4\tau_0)\cosh^2(\pi t/2\tau_0)}\right]^x, \tag{13}
$$

where we have used $\cos(x+\pi) = -\cos(x)$ and $\sin(x+\frac{\pi}{2}) = \cos(x)$.

When we take $r \gg \tau_0$ and $t \gg \tau_0$ the two-point function becomes of the following simplified form

$$
\langle\Phi(r,t)\Phi(0,t)\rangle_{\text{strip}} = \left(\frac{\pi}{2\tau_0}\right)^{2x} \left(\frac{e^{\frac{\pi r}{2\tau_0}} + e^{\frac{\pi t}{2\tau_0}}}{e^{\frac{\pi r}{2\tau_0}} \cdot e^{\frac{\pi t}{2\tau_0}}}\right)^x. \tag{14}
$$

We can approximate the behavior of this two-point function by looking at two different time intervals, namely $t < r/2$ and $t < r/2$. We then see that

$$
\langle\Phi(r,t)\Phi(0,t)\rangle_{\text{strip}} \propto \begin{cases} e^{\frac{-x\pi t}{\tau_0}} & \text{for } t < r/2 \\ e^{\frac{-x\pi r}{2\tau_0}} & \text{for } t > r/2 \end{cases} \tag{15}
$$

Thus, as can be seen from this formula, at fixed r the two-point function decays exponentially in time up to t equals $r/2$ and then doesn't depend on time anymore, i.e. stays constant in time, but only depends exponentially on the separation r.

In the case of fixed initial conditions, with one-point function given by equation 7, we have to consider the connected correlation function. The connected correlation function is the full correlation function minus the disconneted parts

$$\langle \Phi(r,t)\Phi(0,t)\rangle_{\text{conn}} = \langle \Phi(r,t)\Phi(0,t)\rangle_{\text{strip}} - \langle \Phi(0,t)\rangle^2, \tag{16}$$

where the squared of the one-point function comes from the fact that the one-point function is not dependent on r, i.e. $\langle \Phi(0,t)\rangle = \langle \Phi(r,t)\rangle$.

Again, by looking at the two different time intervals $t < r/2$ and $t > r/2$ we can approximate the behavior of this function. We then see that

$$\langle \Phi(r,t)\Phi(0,t)\rangle_{\text{conn}} \propto \begin{cases} 0 & \text{for } t < r/2 \\ e^{\frac{-x\pi r}{2\tau_0}} - e^{\frac{-x\pi t}{\tau_0}} & \text{for } t > r/2 \end{cases} \tag{17}$$

When t equals $r/2$ the correlations start evolving exponentially with time and when $t \gg \tau_0$ the correlations take its asymptotic value depending exponentially on the separation.
In the case of disordered initial conditions ($\psi_0(r) = 0$), and thus $\langle \Phi(t)\rangle = 0$ for all times. This leads to the property that full correlation functions are equal to the connected ones.

3.2.2 The Ising Model

Now we'll consider the two-point function for the Ising model. In this case, the two-point function in the UHP is given by

$$\langle \Phi(z_1)\Phi(z_2)\rangle_{\text{UHP}} = \left(\frac{z_{1\bar{2}}z_{2\bar{1}}}{z_{12}z_{\bar{1}\bar{2}}z_{1\bar{1}}z_{2\bar{2}}}\right)^{1/8} F(\eta),$$

with

$$F(\eta) = \frac{\sqrt{1+\eta^{1/2}} \pm \sqrt{1-\eta^{1/2}}}{\sqrt{2}}$$

and

$$\eta = \frac{z_{1\bar{1}}z_{2\bar{2}}}{z_{1\bar{2}}z_{2\bar{1}}}.$$

The sign $\pm$ in the definition of $F(\eta)$ depends on the boundary conditions. If we consider fixed boundary conditions, we get a + and if in the case of disordered boundary conditions we get a $-$. To calculate η, we use equation 11 and 10 to get

$$\eta = \frac{2\sin^2(\pi r/2\tau_0)}{\cosh(\pi r/2\tau_0) - \cos(\pi\tau/\tau_0)}. \tag{18}$$

Using the results from the gaussian case, we obtain

$$\langle\Phi(r,\tau)\Phi(0,\tau)\rangle_{\text{strip}} = \frac{1}{\sqrt{2}}\left[\left(\frac{\pi}{2\tau_0}\right)^2 \frac{\cosh(\pi r/2\tau_0) - \cos(\pi\tau/\tau_0)}{2\sinh^2(\pi r/4\tau_0)\sin^2(\pi\tau/2\tau_0)}\right]^{1/8}$$

$$\cdot\left[\sqrt{1 + \frac{\sqrt{2}\sin(\pi r/2\tau_0)}{\sqrt{\cosh(\pi r/2\tau_0) - \cos(\pi\tau/\tau_0)}}} \pm \sqrt{1 - \frac{\sqrt{2}\sin(\pi r/2\tau_0)}{\sqrt{\cosh(\pi r/2\tau_0) - \cos(\pi\tau/\tau_0)}}}\right].$$

We analytically continue to real time by $\tau = \tau_0 + it$ and get

$$\langle\Phi(r,\tau)\Phi(0,\tau)\rangle_{\text{strip}} = \frac{1}{\sqrt{2}}\left[\left(\frac{\pi}{2\tau_0}\right)^2 \frac{\cosh(\pi r/2\tau_0) + \cosh(\pi t/\tau_0)}{2\sinh^2(\pi r/2\tau_0)\cosh^2(\pi t/2\tau_0)}\right]^{1/8}$$

$$\cdot\left[\sqrt{1 + \frac{\sqrt{2}\cosh(\pi r/2\tau_0)}{\sqrt{\cosh(\pi r/2\tau_0) + \cosh(\pi t/\tau_0)}}} \pm \sqrt{1 - \frac{\sqrt{2}\cosh(\pi r/2\tau_0)}{\sqrt{\cosh(\pi r/2\tau_0) + \cosh(\pi t/\tau_0)}}}\right],$$

where we used equation 6 and similar identities. We take r/τ_0 and t/τ_0 again to be much larger than 1, and we get a simplified version:

$$\langle\Phi(r,\tau)\Phi(0,\tau)\rangle_{\text{strip}} = \left(\frac{\pi}{2\tau_0}\right)^{1/4} \frac{1}{\sqrt{2}}\left(\frac{e^{\pi r/2\tau_0} + e^{\pi t/\tau_0}}{e^{\pi r/2\tau_0}e^{\pi t/\tau_0}}\right)^{1/8}$$

$$\cdot\left[\sqrt{1 + \frac{e^{\pi t/2\tau_0}}{\sqrt{e^{\pi r/2\tau_0} + e^{\pi t/\tau_0}}}} \pm \sqrt{1 - \frac{e^{\pi t/2\tau_0}}{\sqrt{e^{\pi r/2\tau_0} + e^{\pi t/\tau_0}}}}\right].$$

For fixed boundary conditions, we take the +-sign and by Taylor expanding the small exponential terms in the square root, we obtain the result for the free boson (up to a factor $\sqrt{2}$), given by equation RESULT OF GAUSSIAN CASE with $x = 1/8$. We get the connected part by subtracting $\langle\Phi(0,t)\rangle^2$ with $A_b^\Phi = 2^{1/4}$ for the Ising model.

$$\langle\Phi(r,t)\Phi(0,t)\rangle_{\text{conn,fix}} = \langle\Phi(r,t)\Phi(0,t)\rangle - \langle\Phi(0,t)\rangle^2$$
$$\simeq \sqrt{2}\left(\frac{\pi}{2\tau_0}\right)^{1/4}\left(\frac{e^{\pi r/2\tau_0} + e^{\pi t/\tau_0}}{e^{\pi r/2\tau_0}e^{\pi t/\tau_0}}\right)^{1/8} - \sqrt{2}\left(\frac{\pi}{2\tau_0}\right)^{1/4} e^{-\pi t/8\tau_0}$$

and so we see that also in the case of the Ising model, the connected correlations develop when $t > r/2$:

$$\langle\Phi(r,t)\Phi(0,t)\rangle_{\text{conn,fix}} \propto \begin{cases} 0 & \text{for } t < r/2 \\ e^{\frac{-\pi r}{16\tau_0}} - e^{\frac{-\pi t}{8\tau_0}} & \text{for } t > r/2 \end{cases}$$

In the case of disordered boundary conditions, the connected correlation function is yet again the same as the full correlation function. We find

$$\langle \Phi(r,t)\Phi(0,t)\rangle_{\text{disorder}} \propto \begin{cases} e^{\frac{-\pi(r-3/2t)}{4\tau_0}} & \text{for } t < r/2 \\ e^{\frac{-\pi r}{16\tau_0}} & \text{for } t > r/2 \end{cases}$$

We see that for a disordered initial condition, we get a space dependence of the correlation function even for $t < r/2$.

3.2.3 The general two-point function

Using the result from the Gaussian model and the Ising model, we will now try to derive some general statements on the form of the two-point function. Specifically, we will want to make comments on the time-dependence on the general correlation functions. To start with, we can say that the two-point function in the UHP has the general form

$$\langle \Phi(z_1)\Phi(z_2)\rangle_{\text{UHP}} = \left(\frac{z_{1\bar{2}}z_{2\bar{1}}}{z_{12}z_{\bar{1}\bar{2}}z_{1\bar{1}}z_{2\bar{2}}}\right)^x F(\eta). \tag{19}$$

The function $F(\eta)$ is, in general, an unknown function that depends on the specifics of the model under consideration. Because we already know how the first part of equation 19 transforms (see equation 9), we only need to investigate $F(\eta)$ for every specific model. Fortunately, we can still make some general statements about $F(\eta)$ by making a few approximations.

Looking back at the investigation of the Ising model, we see that if we take $t, r \gg \tau_0$ in equation 18, we get the following form for η:

$$\eta \sim \frac{e^{\pi t/\tau_0}}{e^{\pi r/2\tau_0} + e^{\pi t/\tau_0}}.$$

For $t < r/2$, we get $\eta \sim e^{\pi(t-r/2)/\tau_0} \ll 1$ and for $t > r/2$, we get $\eta \sim 1$. Therefore, in order to review the asymptotic time behaviour of the correlation function, we only need to consider two cases. For the behaviour deep in the bulk, we have small r and hence $t > r/2$ and $\eta \sim 1$. For the behaviour close to the surface we have big r so $t < r/2$ and $\eta \sim 0$. The corresponding limits for $F(\eta)$ are exactly known and given by [3]:

- for two points deep in the bulk: $F(1) = 1$
- for two points near the surface: $F(\eta) \simeq (A_b^\Phi)^2\eta^{x_b}$ with x_b the boundary scaling dimension of the leading boundary operator to which Φ couples and A_b^Φ the coefficient introduced in equation 5.

Using the result 15, we can then make the following statements about the time-dependence of the two-point function:

$$\langle \Phi(r,t)\Phi(0,t)\rangle \propto \begin{cases} (A_b^\Phi)^2 e^{-x\pi t/\tau_0} e^{\pi x_b(t-r/2)/\tau_0} & \text{for } t < r/2 \\ e^{-x\pi r/2\tau_0} & \text{for } t > r/2 \end{cases}$$

Note that this result is useful, but one should note that the sull analytic structure of the CFT is only obtained by doing a full calculation such as we did for the Gaussian model and the Ising model. Moreover, the behaviour close to $t = r/2$ can only be analysed by using the detailed form of $F(\eta)$ depending on the model.

3.3 Correlation functions at different times

In the previous sections we have considered the correlation functions at equal times, so the natural step to take is to consider the two-point functions at different real times $\langle \Phi(r,t)\Phi(0,s)\rangle$. Again, following the same approach as for the equal times, this is achieved by mapping the imaginary time strip to the UHP, but this times the two points are $w_1 = r + i\tau_1$ and $w_2 = 0 + i\tau_2$. We have to go to real time $\tau_1 = \tau_0 + it$ and $\tau_2 = \tau_0 + is$ at the end of the calculation. This map is shown in Figure 2, where $\theta_1 = \frac{\pi\tau_1}{2\tau_0}$ and $\theta_2 = \frac{\pi\tau_2}{2\tau_0}$.

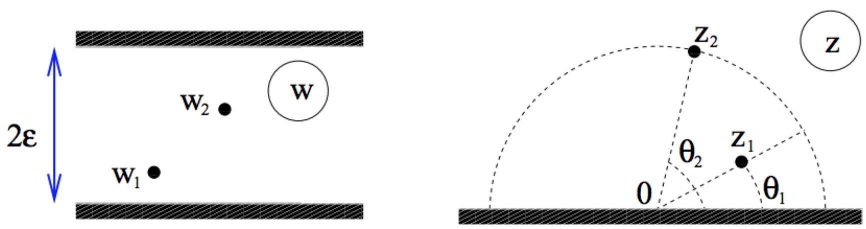

Figure 2: *Left:* Infinite imaginary time strip for points of the correlation functions at different times. *Right:* Conformal mapping of the infinite time strip to the upper-half plane (c.f. Eq. 3), where $\theta_i = \pi\tau_i/2\tau_0$.

Similarly, first we have to calculate the distances z_{ij} in order to solve equation 8.

$$
\begin{aligned}
z_{1\bar{2}}z_{2\bar{1}} &= |z_1 - z_{\bar{2}}||z_2 - z_{\bar{1}}| \\
&= \left|e^{\pi r/2\tau_0}e^{i\pi\tau_1/2\tau_0} - e^{-i\pi\tau_2/2\tau_0}\right|\left|e^{i\pi\tau_2/2\tau_0} - e^{\pi r/2\tau_0}e^{-i\pi\tau_1/2\tau_0}\right| \\
&= 1 + e^{\pi r/\tau_0} - 2e^{\pi r/2\tau_0}\cos(\theta_1 + \theta_2) \\
&= e^{\pi r/2\tau_0}\left(e^{-\pi r/2\tau_0} + e^{\pi r/2\tau_0} - 2\cos(\theta_1 + \theta_2)\right) \\
&= 2e^{\pi r/2\tau_0}\left(\cosh(\pi r/2\tau_0) - \cos(\theta_1 + \theta_2)\right),
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
z_{1\bar{1}}z_{2\bar{2}} &= |z_1 - z_{\bar{1}}||z_2 - z_{\bar{2}}| \\
&= e^{\pi r/2\tau_0}\left|e^{i\pi\tau_1/2\tau_0} - e^{-i\pi\tau_1/2\tau_0}\right|\left|e^{i\pi\tau_2/2\tau_0} - e^{-i\pi\tau_2/2\tau_0}\right| \\
&= 4e^{\pi r/2\tau_0}\sin(\theta_1)\sin(\theta_2)
\end{aligned} \tag{21}
$$

and

$$
\begin{aligned}
z_{12}z_{\bar{1}\bar{2}} &= |z_1 - z_2||z_{\bar{1}} - z_{\bar{2}}| \\
&= \left|e^{\pi r/2\tau_0}e^{i\pi\tau_1/2\tau_0} - e^{i\pi\tau_2/2\tau_0}\right|\left|e^{\pi r/2\tau_0}e^{-i\pi\tau_1/2\tau_0} - e^{-i\pi\tau_2/2\tau_0}\right| \\
&= 1 + e^{\frac{\pi r}{\tau_0}} - 2e^{\frac{\pi r}{2\tau_0}}\cos(\theta_1 - \theta_2) \\
&= e^{\pi r/2\tau_0}\left(e^{-\pi r/2\tau_0} + e^{\pi r/2\tau_0} - 2\cos(\theta_1 - \theta_2)\right) \\
&= 2e^{\pi r/2\tau_0}\left(\cosh(\pi r/2\tau_0) - \cos(\theta_1 - \theta_2)\right).
\end{aligned} \tag{22}
$$

Hence, the two-point function on the strip is given by

$$
\langle\Phi(r,\tau_1)\Phi(0,\tau_2)\rangle_{\text{strip}} = \left|\frac{dw(z_1)}{dz_1}\right|^{-x}\left|\frac{dw(z_2)}{dz_2}\right|^{-x}\langle\Phi(z_1(w))\Phi(z_2(w))\rangle_{\text{UHP}}
$$
$$
= \left[\left(\frac{\pi}{2\tau_0}\right)^2 \frac{\cosh(\pi r/2\tau_0) - \cos(\pi(\tau_1+\tau_2)/2\tau_0)}{4\sin(\pi\tau_1/2\tau_0)\sin(\pi\tau_2/2\tau_0)(\cosh(\pi r/2\tau_0) - \cos(\pi(\tau_1-\tau_2)/2\tau_0))}\right]^x, \tag{23}
$$

where for $\tau_1 = \tau_2$ this formula reduces back to equation 9, as was expected. If we analytically continue to real time by $\tau_1 = \tau_0 + it$ and $\tau_2 = \tau_0 + is$ we get

$$
\begin{aligned}
\langle\Phi(r,t)\Phi(0,s)\rangle_{\text{strip}} &= \left[\left(\frac{\pi}{2\tau_0}\right)^2 \frac{\cosh(\pi r/2\tau_0) + \cosh(\pi(t+s)/2\tau_0)}{4\cosh(\pi t/2\tau_0)\cosh(\pi s/2\tau_0)(\cosh(\pi r/2\tau_0) - \cosh(\pi(t-s)/2\tau_0))}\right]^x \\
&= \left[\left(\frac{\pi}{2\tau_0}\right)^2 \frac{\cosh(\pi r/2\tau_0) + \cosh(\pi(t+s)/2\tau_0)}{2(\cosh(\pi(t-s)/2\tau_0) + \cosh(\pi(t+s)/2\tau_0))(\cosh(\pi r/2\tau_0) - \cosh(\pi(t-s)/2\tau_0))}\right]^x,
\end{aligned} \tag{24}
$$

where in the last line we have used that $\cosh(x)\cosh(y) = \frac{1}{1}(\cosh(x-s) + \cosh(x+y)$. Next, if we consider $r, t, s, |t-s| \gg \tau_0$ we obtain

$$\langle\Phi(r,t)\Phi(0,s)\rangle_{\text{strip}} = \left(\frac{\pi}{2\tau_0}\right)^{2x}\left(\frac{e^{\frac{\pi r}{2\tau_0}}+e^{\frac{\pi t}{2\tau_0}}}{e^{\frac{\pi(t+s)}{2\tau_0}}\cdot(e^{\frac{\pi r}{2\tau_0}}+e^{\frac{\pi|t-s|}{2\tau_0}})}\right)^{x}. \tag{25}$$

We can approximate the behavior of this correlation function for three different regions

$$\langle\Phi(r,t)\Phi(0,t)\rangle_{\text{strip}} \propto \begin{cases} e^{\frac{-x\pi(t+s)}{4\tau_0}} & \text{for } r > t+s \\ e^{\frac{-x\pi r}{4\tau_0}} & \text{for } t-s < r < t+s \\ e^{\frac{-x\pi|t-s|}{4\tau_0}} & \text{for } r < |t-s| \end{cases} \tag{26}$$

We can generalize this two-point function to the most general CFT following the procedure discussed in the subsection above. Again we distinguish two different cases, namely a theory with fixed initial conditions (i.e., $\langle\Phi\rangle \neq 0$) and one with disordered initial conditions (i.e., $\langle\Phi\rangle = 0$). In the first case only the crossover points are changed by the expression for $F(\eta)$, leaving the asymptotic results for the other points unchanged. In the second case, only the first region (i.e., $r > t+s$) picks up an extra factor of $e^{-\pi x_b(t+s-r)/4\tau_0}$.

3.4 Evolution with boundaries

A final modification we will consider in this section is adding a boundary condition at $r = 0$. For simplicity, we will assume that this boundary condition is the same as the initial boundary condition. An example would be, for an Ising-like system, to fix all the spins at $t = 0$ and at the boundary $r = 0$ (for all t) to point in the same direction. The spacetime region of this CFT is depicted in Figure 3.

We will define a new map into the UHP, given by

$$z(w) = \sin\left(\frac{\pi w}{2\tau_0}\right) \tag{27}$$

with $w = \tau + ir$ such that the corners, lying now at $/\pm\tau_0$ (instead of $\tau = 0, 2\tau_0$)

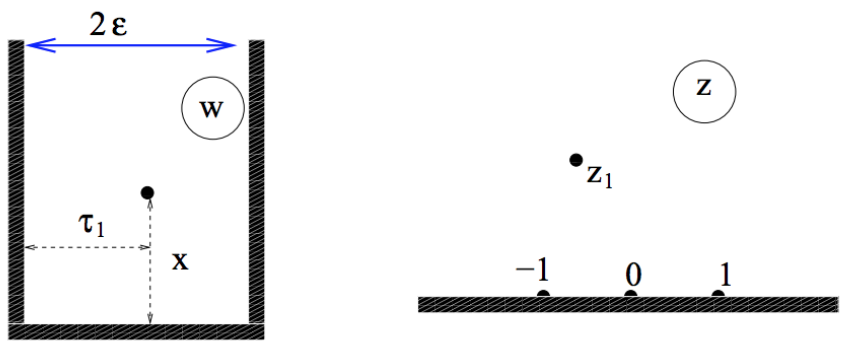

Figure 3: *Left:* The space-time region for the one-point function in a boundary (at $r = 0$ geometry. Note that $w = \tau + ir$. *Right:* Conformal mapping to the upper-half plane, c.f. Eq. 27).

are mapped to $z = \pm 1$. With this new mapping, w_1 becomes

$$\begin{aligned} z_1 \equiv z(w_1) &= z(\tau_1 - \tau_0 + ir) \\ &= \sin\left(-\pi/2 + \pi\tau_1/2\tau_0 + i\pi r/2\tau_0\right) \\ &= -\cos\left(\pi\tau_1/2\tau_0 + i\pi r/2\tau_0\right) \\ &= \frac{1}{2}\left(-e^{i\pi\tau_1/2\tau_0}e^{-\pi r/2\tau_0} - e^{-i\pi\tau_1/2\tau_0}e^{\pi r/2\tau_0}\right) \\ &= \frac{1}{4}(-e^{i\pi\tau_1/2\tau_0}e^{\pi r/2\tau_0} - e^{i\pi\tau_1/2\tau_0}e^{-\pi r/2\tau_0} \\ &- e^{-i\pi\tau_1/2\tau_0}e^{\pi r/2\tau_0} - e^{-i\pi\tau_1/2\tau_0}e^{-\pi r/2\tau_0} \\ &+ e^{i\pi\tau_1/2\tau_0}e^{\pi r/2\tau_0} - e^{i\pi\tau_1/2\tau_0}e^{-\pi r/2\tau_0} \\ &- e^{-i\pi\tau_1/2\tau_0}e^{\pi r/2\tau_0} + e^{-i\pi\tau_1/2\tau_0}e^{-\pi r/2\tau_0}) \\ &= -\cos\left(\pi\tau_1/2\tau_0\right)\cosh(\pi r/2\tau_0) + i\sin(\pi\tau_1/2\tau_0)\sinh(\pi r/2\tau_0). \end{aligned}$$

In this representation, it is now easy to determine the one-point function in the UHP.

$$\langle\Phi(z_1)\rangle_{\text{UHP}} \propto |\text{Im}\, z_1|^{-x} \to [\sin(\pi\tau_1/2\tau_0)\sinh(\pi r/2\tau_0)]^{-x}\,.$$

We also observe that

$$|w'(z_1)|^2 = \left(\frac{2\tau_0}{\pi}\right)^2 \frac{1}{|1 - z_1^2|} \propto \frac{1}{\cosh(\pi r/\tau_0) - \cos(\pi\tau_1/\tau_0)}.$$

Then the one-point function on the strip becomes

$$\langle\Phi(w_1)\rangle_{\text{strip}} = |w'(z_1)|^{-x}\langle\Phi(z_1)\rangle_{\text{UHP}} \propto \left[\frac{sin^2(\pi\tau_1/2\tau_0)\sinh^2(\pi r/2\tau_0)}{\cosh(\pi r/\tau_0) - \cos(\pi\tau_1/\tau_0)}\right]^{-x/2}.$$

We can now continue to real time by taking $\tau_1 = it$ and get

$$\langle \Phi(t,r) \rangle \propto \left[\frac{\cosh(\pi r/\tau_0) + \cosh(\pi t/\tau_0)}{\cosh^2(\pi t/2\tau_0) \sinh^2(\pi r/2\tau_0)} \right]^{x/2}$$

which, taking $t, r \gg \tau_0$, can be approximated as

$$\langle \Phi(t,r) \rangle \propto \left[\frac{e^{\pi r/\tau_0} + e^{\pi t/\tau_0}}{e^{\pi r \tau_0} e^{\pi t/\tau_0}} \right]^{x/2} = \begin{cases} e^{-\pi x t/2\tau_0} & \text{for } t < r \\ e^{-\pi x r/2\tau_0} & \text{for } t > r \end{cases}$$

We should note that in this case, the characteristic time of the 'horizon' is not $t = r/2$ but $t = r$.

4 Characteristics of the 1+1D CFT results

We will now try to draw some general conclusions from the stud the study performed in the previous chapters. In order to investigate how the the state $|\psi(t)\rangle = e^{-iHt}|\psi_0\rangle$ evolves in the thermodynamic limit, we considered the simpler question of how correlation functions evolve and whether they reach constant values for very large times. We studied in general the quench dynamics of one- and two-point functions in a 1+1 CFT. The results of our study rely on the assumption that the leading asymptotic behavior, i.e. large imaginary times, given by 1+1 CFT, may simply be analytically continued to real time to find the characteristics at large real time. Making this assumption, we found for the one-point function of a scalar primary field that, when taking the thermodynamic limit, it decays to the ground-state value exponentially in time. For the two-point functions, we can observe two general features. First, at $t = r/2$ there is a sharp horizon (or light-cone) effect due to the fact that the behavior before and after this t is totally different. Mostly, if we consider the full correlation function, for one region this function depends exponentially on t, whereas for the other region it does not depend on t but only on the separation r. Second, we discovered that connected two-point functions of primary operators at separation r vanish for $t < r/2$, whereas they arrive exponentially fast at a constant value depending exponentially on the distance r for $t > r/2$.

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