Zamolodchikov's c-theorem & Cardy's a-theorem

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1 Introduction

In this summary we will first discuss general background on CFT and some prerequisites for understanding Zamolodchikov's c-theorem, After that we will give a prove of the c - theorem and give an application of the c - theorem. Then we will discuss Cardy's a-theorem and end with the current status of the a-theorem.

1.1 Definitions

$$\partial_a = \frac{\partial}{\partial g^a} \tag{1}$$

$$\dot{a} = \frac{\partial}{\partial t} a \tag{2}$$

Within expectation value signs:

$$A^1 \equiv A_1(x_1) \tag{3}$$

1.2 Background on CFT

A conformal field theory (CFT) is a field theory which is invariant under conformal transformation. Conformal transformations are coordinate transformations which leave the metric tensor invariant up to a scale factor $\Omega(x)$.

$$\begin{aligned} &x^a \to x'^a(x) \\ &g_{\alpha\beta} \to g_{\alpha'\beta'}(x') = \Omega(x) g_{\alpha\beta}(x) \end{aligned} \tag{4}$$

The physical nature of this invariance depends on the theory considered. If the metric is fixed the conformal symmetry is a real physical symmetry, if the metric is dynamical then the symmetry is a gauge symmetry. To find the constraints on the coordinate transformation we calculate the change of the metric. Under infinitesimal general coordinate transformation $x^\mu \to x'^\mu(x) = x^\mu(x) + \epsilon^\mu(x)$ the metric changes as,

$$\begin{aligned} g_{\rho'\sigma'}(x) &= \frac{dx^\mu}{dx^{\rho'}} \frac{dx^\nu}{dx^{\sigma'}} g_{\mu\nu}(x) \\ &= (\delta^\mu_{\rho'} - \partial_{\rho'} \epsilon^\mu(x))(\delta^\nu_{\sigma'} - \partial_{\sigma'} \epsilon^\nu(x)) g_{\mu\nu}(x) = \\ &= g_{\rho'\sigma'}(x) - \partial_{\rho'} \epsilon_{\sigma'}(x) - \partial_{\sigma'} \epsilon_{\rho'}(x) + O(\epsilon^2) \\ &= g_{\rho'\sigma'}(x) - f(x) g_{\rho'\sigma'}(x), \qquad \partial_{\rho'} \epsilon_{\sigma'}(x) + \partial_{\sigma'} \epsilon_{\rho'}(x) = f(x) g_{\rho'\sigma'}(x) \end{aligned} \tag{5}$$

where the RHS of the last line contains the constraints on $\epsilon(x)$.

Veryfing that the last line of eq. (5) is indeed conformal means that is must be of the form of eq. (4) ,

$$\begin{aligned} &g_{\rho'\sigma'}(x) - f(x) g_{\rho'\sigma'}(x) = \\ &\Omega(x) g_{\rho'\sigma'}(x), \end{aligned} \tag{6}$$

with $\Omega(x) = [1 - f(x)]$.

Contracting the RHS of eq. (5) with $\delta^{\rho'\sigma'}$ and assuming the flat metric, leads to $f(x) = \frac{2}{d}\partial \cdot \epsilon$ and find $\delta^{\rho'\sigma'}[\partial_{\rho'}\epsilon_{\sigma'}(x) + \partial_{\sigma'}\epsilon_{\rho'}(x)] = \frac{2}{d}\partial \cdot \epsilon$. To find the most general solution to this equation we consider,

$$[\delta_{\rho\sigma} + (d-2)\partial_\rho\partial_\sigma]\partial \cdot \epsilon = 0 \tag{7}$$

which is obtained after applying two derivatives and contracting one pair of indices. Solutions to eq. (7) are at most quadratic in x if $d > 2$ [8]. One immediately sees that $d = 2$ gives special properties to ϵ which are discussed in the upcoming chapters.

The solutions are at most quadratic in x imply that (infinitesimal) translation $\epsilon^\rho = a^\rho$, scaling $\epsilon^\rho = \lambda x^\rho$, rotation $\epsilon^\rho = m^\rho_\sigma x^\sigma$ and the special conformal transformation $\epsilon^\rho = b^\rho x^2 - 2x^\rho b \cdot x$ are part of the conformal transformations.

1.3 Core properties of a d-dimensional CFT

The stress-energy tensor in classical flat space (fixed metric) is found when varying the action with a globally defined $\delta x^u = \epsilon^u$, thus making use of translational invariance. But there is a quicker way to determine the stress-energy tensor of a field theory. By often demanding our theory to be invariant under the local variation $\delta x^u = \epsilon^u(x)$ [9]. The variation of the action then takes the form,

$$\delta S = \int d^d x J^u(x) \partial_u \epsilon(x) + ME \tag{8}$$

if ϵ is globally defined then 8 is obviously zero and we have found a symmetry of the theory. With epsilon now depending on the spacetime coordinates we use partial integration to find that when our matter field equations (ME) are on shell $\partial_\alpha J^\alpha$ must be zero to have a symmetry.

Tong uses [9] the fact that if we now consider a general dynamical metric then we can view the coordinate transformation $\delta x^u = \epsilon^u(x)$ as a symmetry of the metric. Knowing that the action is invariant if one makes the change $\delta g_{ab} = \partial_\alpha \epsilon_\beta + \partial_\beta \epsilon_\alpha$, as proved in (5), to the metric. Such that the two transformations cancel each other.

This knowledge can be exploited by realizing that if we only transform the metric in this way (5) we get the opposite varation when varying the action with $\delta x^u = \epsilon^u(x)$. So under our coordinate transformation our action changes as,

$$\begin{aligned}
\delta S &= -\int d^d x \frac{\delta S}{\delta g_{\alpha\beta}} \delta g_{\alpha\beta} \\
\delta S &= -\int d^d x \frac{\delta S}{\delta g_{\alpha\beta}} (\partial_\alpha \epsilon_\beta + \partial_\beta \epsilon_\alpha) \\
\delta S &= -2 \int d^d x \frac{\delta S}{\delta g_{\alpha\beta}} \partial_\alpha \epsilon_\beta \\
\delta S &= \frac{1}{2\pi} \int d^d x \sqrt{g} (T^{\alpha\beta} \partial_\alpha \epsilon_\beta), \qquad T^{\alpha\beta} = -\frac{4\pi}{\sqrt{g}} \frac{\delta S}{\delta g_{\alpha\beta}} \\
\delta S &= \frac{1}{2\pi} \int d^d x \sqrt{g} (\partial_\alpha T^{\alpha\beta}) \epsilon_\beta = 0
\end{aligned} \tag{9}$$

where we ignored the boundary term in the last line. Concluding that $\partial_\alpha T^{\alpha\beta} = 0$, which ensures the conservation of the stress-energy tensor.

Particularly interesting in CFT is the coordinate change related to scaling $\delta x^u = \epsilon^u(x) = \lambda x^u$. Easily calculating the change of the metric as,

$$\begin{aligned}
\delta g_{\alpha\beta} &= \lambda \partial_\alpha x_\beta + \lambda \partial_\beta x_\alpha = 2\lambda \delta_{\alpha\beta} \\
\delta S &= -2 \int d^d x \frac{\delta S}{\delta g_{\alpha\beta}} \lambda \delta_{\alpha\beta} \\
\delta S &= 2\lambda \int d^d x \frac{\sqrt{g}}{4\pi} T^{\alpha\beta} \delta_{\alpha\beta}
\end{aligned} \tag{10}$$

demanding that the action vanishes we conclude that $T^{\alpha\beta}\delta_{\alpha\beta} = T^{\alpha}_{\alpha} = 0$, the stress-energy tensor vanishes in a classical CFT. This statement will also hold in the quantum regime in flat spacetime.

In quantum curved space we get that the $\langle T^{\alpha}_{\alpha}\rangle = -\frac{c}{12}R$ where R is the ricci scalar and c is the central charge. The central charge is equal to the number of degrees of freedom. For example a single free boson has $c = 1$, a free fermion has $c = \frac{1}{2}$.

1.4 2D CFT in complex coordinates

To get to the core of CFT we will only consider 2D(x^0,x^1) CFT in flat space for now. The reason why is that any analytic change of coordinates will result in a conformal transformation. We define the complex coordinates $z = x^0 + ix^1$ and $\bar{z} = x^0 - ix^1$ such that under this coordinate transformation our diagonal metric $\delta_{\mu\nu}$ changes to,

$$
\begin{aligned}
x^0 &= \frac{1}{2}(z + \bar{z}) \\
x^1 &= \frac{1}{2}(z - \bar{z}) \\
\rho, \sigma &= z, \bar{z} \\
g_{\rho\sigma}(x) &= \frac{dx^\mu}{dx^\rho}\frac{dx^\nu}{dx^\sigma}\delta_{\mu\nu} = \frac{dx^0}{dx^\rho}\frac{dx^0}{dx^\sigma} + \frac{dx^1}{dx^\rho}\frac{dx^1}{dx^\sigma} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}
\end{aligned}
\tag{11}
$$

Continuing with our new metric and verifying that analytic coordinate transformations are conformal transformations,

$$
\begin{aligned}
z &\to z' = f(z) \\
\bar{z} &\to \bar{z}' = \bar{f}(\bar{z}) \\
g_{\mu\nu}(x) &= \frac{dx^\rho}{dx^\mu}\frac{dx^\sigma}{dx^\nu}g_{\rho\sigma}(x) = \frac{dz}{dx^\rho}\frac{d\bar{z}}{dx^\sigma} = \frac{dz}{df(z)}\frac{d\bar{z}}{d\bar{f}(\bar{z})}
\end{aligned}
\tag{12}
$$

such that

$$
ds^2 = dzd\bar{z} \to \left|\frac{df}{dz}\right|^2 dzd\bar{z} \tag{13}
$$

which satisfies our conformal transformation defined in eq. (4).

1.5 Properties of stress-energy tensor in complex coordinates

Considering our previously chosen complex coordinates we get some interesting propeties of the stress energy tensor. Starting with the conservation of the stress-energy tensor,

$$\partial_\alpha T^{\alpha\beta} = \partial_z T^{z\beta} + \partial_{\bar{z}} T^{\bar{z}\beta} \tag{14}$$

$$\partial_\alpha T^{\alpha z} = \partial_z T^{zz} + \partial_{\bar{z}} T^{\bar{z}z} = \partial_z T^{zz} = 0 \tag{15}$$

$$\partial_\alpha T^{\alpha\bar{z}} = \partial_z T^{z\bar{z}} + \partial_{\bar{z}} T^{\bar{z}\bar{z}} = \partial_{\bar{z}} T^{\bar{z}\bar{z}} = 0 \tag{16}$$

lowering the indices on the stress-energy tensor leads to $\partial_{\bar{z}} T_{zz} = 0$ and $\partial_z T_{\bar{z}\bar{z}} = 0$. Hence T_{zz} is analytic and $T_{\bar{z}\bar{z}}$ is anti-analytic. Using similar arguments we can calculate the conserved current related to the analytic variation $z' \to z+\epsilon(z)$ with conserved currents $J_{\bar{z}} = 0$ and $J_z = T_{zz}(z)\epsilon(z)$. For anti-analytic variation $\bar{z'} \to \bar{z} + \bar{\epsilon}(\bar{z})$ with conserved currents $\bar{J}_z = 0$ and $\bar{J}_{\bar{z}} = \bar{T}_{\bar{z}\bar{z}}(\bar{z})\epsilon(\bar{z})$. Hence the analytic variation generates an analytic current and the anti-analytic varation generates an anti-analytic current. Notice that we can also apply both variations to create a mixture of analytic/anti-analytic currents.

Moving on to the constraint on the stress-energy tensor originating from the traceless property

$$T^{\alpha\beta} g_{\alpha\beta} = T^{z\bar{z}} g_{z\bar{z}} + T^{\bar{z}z} g_{\bar{z}z} = T^{z\bar{z}} = 0. \tag{17}$$

1.6 Callan-Symanzik equation

In the text that follows, we consider an arbitrary action functional $S[\phi]$, with a corresponding Lagrangian $\mathcal{L}(x)$ such that $S[\phi] = \int d^2x \mathcal{L}[\phi(x)]$ is obeyed, and with ϕ a field. Consider also g, a set of variables that are coefficients associated to each order of operator. Space $\mathcal{S}$ is the space spanned by these variables g. Integrating the action to get the correlator, partition function and other related functions is generally done up until a "UV-cutoff", the highest energy at which the theory is still valid. Correspondingly, only the operators in the Lagrangian which are relevant below the cut-off are considered, and any sub-leading orders of the field ϕ are truncated.

Flowing from one theory to the other by varying this UV-cutoff is the central idea of renormalization. Varying the UV cut-off corresponds to the idea of moving through $\mathcal{S}$. In this interpretation, taking the derivative with respect to the coefficients in set $g = (g^1, g^2, \dots)$ gives us basis vectors of the theory, which we will denote by

$$\phi_i(x) \equiv \frac{\partial \mathcal{L}[\phi(x)]}{\partial g^i}, \tag{18}$$

which is identical to saying that

$$\mathcal{L} = g_i \phi^i \tag{19}$$

is a decomposition of the Lagrangian.

Commonly it is assumed that we have some definite form of Lagrangian, and work with the theory from there. We define the variations ourselves, define the Lagrangian, and see what the theory describes. In CFT, another method is available: using restrictions from CFT to define what forms the formulas can have in the first place (the *conformal bootstrap*).

Suppose we work in a theory where the expectation values are linked by the following equality (assume $x \neq x_i$ so that all correlation functions are convergent):

$$\left\langle A^1 \cdots A^N \right\rangle = \int \mathcal{D}\phi A^1 \cdots A^N \exp\left(-S[\phi]\right) \tag{20}$$

with S the usual action and A^k operators in the field theory. Using the variation $\delta S = (2\pi)^{-1} \int d^2x \partial_\mu \epsilon_\nu T^{\mu\nu}(x)$ as derived in the Student Seminar lectures, this leads to the Ward identity,

$$\sum_{i=1}^{N} \left\langle A^1 \cdots \delta A^i \cdots A^N \right\rangle = (2\pi)^{-1} \int d^2x \partial_\mu \epsilon_\nu \left\langle T^{\mu\nu}(x) A^1 \cdots A^N \right\rangle. \tag{21}$$

Note that we have used the particular variation $\delta x^\mu = \epsilon^\mu$ to derive this.

Adding a term of the form of the continuity equation, $(2\pi)^{-1} \int d^2x \partial_\mu \left\langle T^{\mu\nu} X \right\rangle$ with X some operator, gives a divergence term:

$$\sum_{i=1}^{N} \left\langle A^1 \cdots \delta A^i \cdots A^N \right\rangle = (2\pi)^{-1} \int d^2x \partial_\mu \left\langle \epsilon_\nu T^{\mu\nu}(x) A^1 \cdots A^N \right\rangle \tag{22}$$

where we have taken the liberty of integrating over the continuity equation itself $\partial_\mu \left\langle T^{\mu\nu} X \right\rangle$. The integral of this zero-valued quantity is here again assumed to be zero.

Zamolodchikov[2] then claims that the variation must therefore take the form[1]

$$\delta A^i = \oint_{\partial \mathcal{A}} dy^\lambda \rho_{\lambda\mu} \epsilon_\nu(y) T^{\mu\nu}(y) A^i + \frac{1}{2\pi} \int_{\mathcal{A}} d^2y \partial_\mu \epsilon_\nu(y) T^{\mu\nu} A^i \tag{23}$$

with $\mathcal{A}$ an arbitrary surface in $\mathbb{R}^2$ and $\rho_{\lambda\mu}$ the completely antisymmetric tensor. Allowing $\mathcal{A}$ to be arbitrarily small, we find the usual definition of the variation of the operator A^i.

A logical next step is to find how an operator A specifically will vary with x. Generally a translation will lead to the variation $\delta A(x) = \epsilon^\mu \partial A(x)/\partial x^\mu$, which follows directly from the passive transformation associated to the variation $\delta x^\mu = \epsilon^\mu$. Similarly, a dilation from the origin can be represented as some operator $\hat{D}$ acting on the field $A(x)$ at $x = 0$. An arbitrary dilation requires translation from a point $x = (x^1, \dots)$ to the origin $x = 0$, such that the arbitrary dilation can be represented as $\delta A(x) = \lambda \left[x^\mu \frac{\partial}{\partial x^\mu} + \hat{D} \right] A(x)$ which manifestly

[1] But we have not been able to derive this specific form of the variation. Our best guess as of now is that the way the trace is defined $\Theta(x) \equiv -T^\mu{}_\mu$, one could obtain a minus sign somewhere, but we could not verify this.

produces the correct result about $x = 0$. Here where we have chosen $\epsilon^\mu = \lambda x^\mu$. This specific choice also has consequences on the RHS of for example eq. (21) and is the common argument to show that the trace of the stress-energy tensor vanishes. In this case eq. (21) gives

$$\sum_{i=1}^{N} \left\langle A^1 \cdots \left[x^\mu \tfrac{\partial}{\partial x^\mu} + \hat{D}_i \right] A^i \cdots A^N \right\rangle = (2\pi)^{-1} \int d^2x \partial_\mu \left(x_\nu \right) \left\langle T^{\mu\nu}(x) A^1 \cdots A^N \right\rangle \tag{24}$$

where, after identifying $\partial_\mu x_\nu \equiv \delta_{\mu\nu}$, we find the trace of T on the RHS.

Zamolodchikov's trick is to consider a similar variation in the cut-off parameter of the field theory, where one looks at how a general operator varies as the RG parameters g^i vary with the cut-off[2]. The idea is that we again define a variation

$$\delta A = \hat{B}_a A = \partial_a A. \tag{25}$$

Identifying $\delta S = \beta^a \phi_a$ or similarly $\theta(x) = 2\pi \beta^a(g) \phi_a(x)$ in the text above (i.e., pick a basis with the β functions as coefficients as outlined in the main text), this additional "symmetry" leads to

$$\sum_{i=1}^{N} \left\langle A^1 \cdots \left[x^\mu \tfrac{\partial}{\partial x^\mu} + \hat{D}_i - \beta^a \partial_a \right] A^i \cdots A^N \right\rangle = (2\pi)^{-1} \int d^2x \beta^a \partial_a \left\langle \phi(x) A^1 \cdots A^N \right\rangle \tag{26}$$

where the minus sign on the LHS comes from the definition of the trace $\theta \equiv -T^\mu_\mu$. The above equation describes the running of the operator A with the energy (implied by the beta functions) and is commonly called the Callan-Symanzik equation. One can define additional symmetries but these turn out to be either obsolete or composed of the symmetries described above, such that the argument below holds identically. Also replacing $\hat{D}_i A^i = \frac{\partial A}{\partial \log a}$ with a the scaling parameter, one recovers equation (8) from Zomolodchikov's paper[1].

2 Zamolochikov's c-theorem

The c-theorem states that in 2 dimensions there is a function $c(g)$ that depends on a coupling constant g which has the following properties:

1. $c(t)$ is a monotonically decreasing function of scale t:

$$\dot{c} = \beta^i(g) \partial_i c(g) \leqslant 0. \tag{27}$$

   At the fixed point $g = g^*$ we have $\beta(g^*) = 0$. β^i is the function that generates the change of the coupling constant g^i.

2. At the fixed point $g = g^*$ we have $\beta(g^*) = 0$ which implies that $\partial_i c = 0$. At this fixed point we have a conformal symmetry.

3. In its fixed point $c(g^*) = c$, where c is the central charge encountered in conformal field theory.

The goal off this digest is to show that a function (called Zomolodchikov's c function) with these properties exists.

3 Definition of Correlators

Define the usual short-hands for the symmetric energy-momentum tensor $T_{\mu\nu}$ in complex coordinates $(z, \bar{z}) \equiv (x^1 + ix^2, x^1 - ix^2)$:

$$T = T_{zz} \tag{28}$$
$$\Theta = -T_{\bar{z}z} \tag{29}$$
$$\bar{T} = T_{\bar{z}\bar{z}}. \tag{30}$$

(Note that in the paper, Zamolodchikov defined Θ without the minus sign [1] but in his book he does use a minus sign [2].)

Also define $\Phi_i(g, x) = \partial_i \sigma(g, a, x)$, where $\sigma(g, a, x)$ is the action density, g the coupling constants, a the UV cut-off scale and x the scaling parameter. The scaling x is defined with respect to some reference action $s(g, a)$, the correlators of which coincide when flowing through space $\mathcal{S}$ along the RG flow path. Space $\mathcal{S}$ is spanned by basis vectors ∂_i from $S(g, a, x)$, and it is defined as $x \equiv e^t a$ $(t > 0)$, where t is some constant which we use to rescale.

By assumption, our couplings g_i run with the energy. Therefore there is a non-zero β-function for every g_i. Thus there exists a basis in which we can expand Θ as follows:

$$\Theta = \beta^i(g)\Phi_i$$

Moreover we see that in a fixed point $\beta^i(g*) = 0$ and thus the trace of the stress energy tensor vanishes. Thus in the fixed point we have a CFT. In a general CFT, the following correlation functions of interest are with $x^2 = z\bar{z}$ [1]:

$$C(g) = 2z^4 \langle T(x)T(0)\rangle |_{x^2=1} \tag{31}$$
$$H_i(g) = z^2 x^2 \langle T(x)\Phi_i(0)\rangle |_{x^2=1} \tag{32}$$
$$G_{ij}(g) = x^4 \langle \Phi_i(x)\Phi_j(0)\rangle |_{x^2=1}. \tag{33}$$

Equivalently one can define, with $t = \log(z\bar{z})$ [2]:

$$\langle T(x)T(0)\rangle = \frac{F(t)}{z^4} \tag{34}$$
$$\langle T(x)\Theta(0)\rangle = \frac{H(t)}{z^3\bar{z}} \tag{35}$$
$$\langle \Theta(x)\Theta(0)\rangle = \frac{G(t)}{z^2\bar{z}^2} \tag{36}$$

Where C, F and G are invariant amplitudes under rotations.

4 Differential equations of invariant amplitudes

Zamalodchikov defines the following c-function

$$c(g) = C(g) + 4\beta^k H_k - 6\beta^i \beta^j G_{ij} \tag{37}$$

and then then takes the derivative:$\beta^i(g)\partial_i$. And uses [1],

$$\frac{1}{2}\beta^k \partial_k C = -3\beta^i H_i + \beta^k \partial_k \left(\beta^i H_i\right). \tag{38}$$

$$\beta^k \partial_k(\beta^i H_i) - \beta^i H_i = \beta^k \beta^k \partial_k(\beta^i \beta^j G_{ij}) - 2\beta^i \beta^j G_{ij}. \tag{39}$$

To be able to prove properties 1-3 of the function $c(g)$, we first show the above relations are true.

Starting from conservation of energy momentum tensor $\partial^\mu T_{\mu\nu} = 0$, we get

$$\partial^\mu \langle T_{\mu\nu} X \rangle = g^{\mu\alpha} \partial_\alpha \langle T_{\mu\nu} X \rangle = g^{\mu z} \partial_z \langle T_{\mu\nu} X \rangle + g^{\mu \bar{z}} \partial_{\bar{z}} \langle T_{\mu\nu} X \rangle = 0 \tag{40}$$

which leads to

$$g^{\bar{z}z} \partial_z \langle T_{\bar{z}\nu} X \rangle + g^{z\bar{z}} \partial_{\bar{z}} \langle T_{z\nu} X \rangle = 0 \qquad \Rightarrow \qquad \partial_z \langle T_{\bar{z}\nu} X \rangle + \partial_{\bar{z}} \langle T_{z\nu} X \rangle = 0. \tag{41}$$

4.1 Equation for $\dot{C}$

In eq. (40), X is some function that does not depend on z, $\bar{z}$ we insert such that the derivative gives no contribution. Because the metric is off-diagonal only $g^{\bar{z}z}$ appears here. Picking $X = T_{zz}(0) = T(0)$ and $\nu = z$, we derive the differential equation

$$\partial_z < T_{\bar{z}z} T(0) > + \partial_{\bar{z}} < T_{zz} T(0) > = \partial_z \left[-\frac{H(t)}{z^3 \bar{z}} \right] + \partial_{\bar{z}} \left[\frac{F(t)}{z^4} \right] \tag{42}$$

$$= \frac{3H(t)}{z^4 \bar{z}} - \frac{\dot{H}(t)}{z^3 \bar{z}} \frac{\partial t}{\partial z} + \frac{\dot{F}(t)}{z^4} \frac{\partial t}{\partial \bar{z}} \tag{43}$$

$$= \frac{1}{z^4 \bar{z}} \left[\dot{F} + 3H - \dot{H} \right] = 0 \tag{44}$$

from which we conclude $\dot{F} = -3H + \dot{H}$. In eq. (42), because of commutativity we can swap $T(0)$ and $T_{\bar{z}z}$. Further, we can pick our coordinate system appropriately so that we can swap the arguments of T and $T_{\bar{z}z}$.

Expanding the derivative d/dt as $\beta^k \partial_k$, and writing $H(t) = \beta^i H_i$ and plugging this into the differential equation, we find the first RG differential equation,

$$\beta^k \partial_k F = \frac{1}{2}\beta^k \partial_k C = -3\beta^i H_i + \beta^k \partial_k \left(\beta^i H_i\right).$$

4.2 Equation for $\dot{H}$

Again starting from our conservation equation with this time $X = T_{z\bar{z}}(0) = \Theta(0)$, but again $\nu = z$, the corresponding equation will be:

$$\partial_z \langle T_{\bar{z}z} T_{z\bar{z}}(0)\rangle + \partial_{\bar{z}} \langle T_{zz} T_{z\bar{z}}(0)\rangle = 0 \tag{45}$$

$$\partial_z \left[\frac{G(t)}{z^2 \bar{z}^2}\right] - \partial_{\bar{z}} \left[\frac{H(t)}{z^3 \bar{z}}\right] = 0 \tag{46}$$

$$\left[-2\frac{G(t)}{z^3 \bar{z}^2} + \frac{\dot{G}(t)}{z^2 \bar{z}^2}\frac{dt}{dz}\right] - \left[\frac{H(t)}{z^3 \bar{z}^2}\right] + \left[\frac{\dot{H}(t)}{z^3 \bar{z}}\frac{dt}{d\bar{z}}\right] = 0 \tag{47}$$

$$z^{-3}\bar{z}^{-2}[-2G + \dot{G} + H - \dot{H}] = 0 \tag{48}$$

$$\dot{H} - H = \dot{G} - 2G. \tag{49}$$

Plugging in $\frac{d}{dt} = \beta^k \partial_k$, $H = \beta^i H_i$ and $G = \beta^i \beta^j G_{ij}$, we get

$$\beta^k \partial_k(\beta^i H_i) - \beta^i H_i = \beta^k \beta^k \partial_k(\beta^i \beta^j G_{ij}) - 2\beta^i \beta^j G_{ij} \tag{50}$$

which confirms eq. (39).

5 Assertion 1

Starting from the defintion of our c-function eq. (37)

$$c(g) = C(g) + 4\beta^k H_k - 6\beta^i \beta^j G_{ij} \tag{51}$$

Where we use:

$$\beta^i \partial_i C(g) = -6\beta^i H_i + 2\beta^k \partial_k(\beta^i H_i) \tag{52}$$

$$\beta^j \beta^k \partial_k G_{ij} + \beta^j (\partial_i \beta^k) G_{jk} + \beta^j (\partial_j \beta^k) G_{ik} = \tag{53}$$

$$\beta^k \partial_k H_i + \partial_i \beta^k H_k - H_i + 2\beta^k G_{ik} \tag{54}$$

which leads to

$$\beta^k \partial_k c(g) = [\beta^k \partial_k C(g) + \beta^k \partial_k(\beta^i H_i)] - 6\beta^k \partial_k(\beta^i \beta^j G_{ij}) \tag{55}$$

$$= [-6\beta^i H_i + 6\beta^k \partial_k(\beta^i H_i)] - 6\beta^k \partial_k(\beta^i \beta^j G_{ij}). \tag{56}$$

Treating the last term with care we obtain

$$\beta^k \partial_k(\beta^i \beta^j G_{ij}) = \tag{57}$$

$$\beta^k (\partial_k \beta^i)\beta^j G_{ij} + \beta^k \beta^k (\partial_k \beta^j) G_{ij} + \beta^k \beta^i \beta^j \partial_k G_{ij} \tag{58}$$

$$\beta^j (\partial_j \beta^k)\beta^i G_{ki} + \beta^i \beta^j (\partial_i \beta^k) G_{jk} + \beta^k \beta^i \beta^j \partial_k G_{ij}. \tag{59}$$

The previous equation contains only dummy indices, so permuting the first and second line after the equals sign as respectively.

$[j \to i; i \to k; k \to j]$
$[j \to k; k \to i; i \to j]$

gives, using (21),

$$\beta^i[\beta^j(\partial_j\beta^k)G_{ki} + \beta^j(\partial_i\beta^k)G_{jk} + \beta^k\beta^j\partial_k G_{ij}] = \quad (60)$$

$$\beta^i[\beta^k\partial_k H_i + \partial_i\beta^k H_k - H_i + 2\beta^k G_{ik}] = \quad (61)$$

$$\beta^i\beta^k\partial_k H_i + \beta^i\partial_i\beta^k H_k - \beta^i H_i + 2\beta^i\beta^k G_{ik} = \quad (62)$$

$$\beta^i\beta^k\partial_k H_i + \beta^k\partial_k\beta^i H_i - \beta^i H_i + 2\beta^i\beta^k G_{ik} = \quad (63)$$

$$\beta^k(\beta^i H_i) - \beta^i H_i + 2\beta^i\beta^k G_{ik} \quad (64)$$

Plugging this into (24) and using that G_{ik} is the metric in the last step:

$$\beta^i\partial_i c(g) = \quad (65)$$

$$[-6\beta^i H_i + 6\beta^k\partial_k(\beta^i H_i)] - 6\beta^k\partial_k(\beta^i\beta^j G_{ij}) = \quad (66)$$

$$[-6\beta^i H_i + 6\beta^k\partial_k(\beta^i H_i)] - 6[\beta^k(\beta^i H_i) - \beta^i H_i + 2\beta^i\beta^k G_{ik}] = \quad (67)$$

$$-12\beta^i\beta^k G_{ik} \quad (68)$$

Where G_{ik} can be thought of as the norm of the state and thus must be positive definite, because of unitarity (Zomolodchikov, p.288). So we can go to a basis where this matrix is diagonal such that we get:

$$-12\beta^i\beta^k G_{ik} = -12[\beta_1^2 G_{11} + \beta_2^2 G_{22} + ...] \quad (69)$$

(70)

6 Assertion 2

Starting from:

$$\beta^i\partial_i c(g) = -12\beta^i\beta^k G_{ik} \quad (71)$$

$$\to \beta^i[\partial_i c(g) + 12\beta^k G_{ik}] = 0 \quad (72)$$

$$\to \partial^k c(g) = -12\beta^k \quad (73)$$

where G^{ik} is some positive definite matrix, the metric in parameter space. So we see that if $\beta^k(g) = 0$, this implies that $\partial_i c(g) = 0$, thus proving assertion 2.

7 Assertion 3

From the definition of $c(g)$ evaluated at $g = g*$, where $\beta^i(g^*) = 0$:

$$c(g) = [C(g) + 4\beta^k(g)H_k - 6\beta^i(g)\beta^j(g)G_{ij}]|_{g=g*} \quad (74)$$

$$c(g^*) = C(g^*) \quad (75)$$

$$c(g^*) = 2z^4[< T(z)T(0) > |_{g=g^*}] = \quad (76)$$

$$2z^4\left[\frac{c/2}{(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} - \frac{\partial_\omega T(\omega)}{z-\omega}\right]|_{\omega=0} \quad (77)$$

$$c(g^*) = c + z^2 T(0) - \partial_\omega T(0) z^3 \quad (78)$$

$$c(g^*) = 2z^4[\frac{c}{2}z^{-4}] = c \quad (79)$$

Where in the last step we used the well known OPE for the energy momentum tensor. And took the limit $z \to 0$.

8 Conclusion

So we see that the value of c decreases when t increases. Therefore if we have two fixed point $g(-\infty) = g_1^*$ and $g(+\infty) = g_2^*$ which are joined by a trajectory $g(t)$ in t-space. Since t has unit of inverse energy squared we conclude that if t gets really large the energy scale is small and when t gets small the energy scale gets large. Then we have the following values for c: $c(g_1^*) = c_1 = c_{UV}$ and $c(g_2^*) = c_2 = c_{IR}$. Where $c_{IR} < c_{UV}$ because c decreases when t increases. Since c represents the number of degrees of freedom, we conclude that at high energies there are more degrees of freedom than at low energies.

9 Applications of the c-theorem

An application of the c-theorem is to calculate how the central charge changes between 2 fixed points. Starting from eq. (68) and using $\beta^i \partial_i = d/dt$:

$$\beta^i \partial_i c = 12 \beta^i \beta^k G_{ik} \tag{80}$$

$$\frac{dc}{dt} = -12 G(t) \tag{81}$$

$$-\int \mathrm{d}c = 12 \int G(t)\ \mathrm{d}t = 12 \int \langle \Theta(x)\Theta(0)\rangle\, (z\bar{z})^2\ \mathrm{d}(\ln(z\bar{z})) = \tag{82}$$

$$12 \int \langle \Theta(x)\Theta(0)\rangle\, r^4\ \mathrm{d}(\ln(r^2)) = 24 \int \langle \Theta(x)\Theta(0)\rangle\, r^3\ \mathrm{d}r \tag{83}$$

$$-\int_{c(0)}^{c(\infty)} \mathrm{d}c = c(0) - c(\infty) = c_1 - c_2 = \Delta c = \tag{84}$$

$$24 \int_0^\infty \langle \Theta(x)\Theta(0)\rangle\, r^3\ \mathrm{d}r \tag{85}$$

Where in eq. (82) we have used eq. (36). And used that $z\bar{z} = r^2$ in and we integrate r from zero (the UV fixed point) to infinity (the IR fixed point). Where after a different normalization one obtains [4]

$$\Delta c = \frac{3}{2} \int_0^\infty \langle \Theta(x)\Theta(0)\rangle\, r^3\ \mathrm{d}r \tag{86}$$

9.1 Application to the Ising model

The purpose of this section is to indicate an application of the c-theorem. To increase readability, we do not rederive results. These can be find in [4]. So one can start with the Ising model action,

$$S^* = \int \psi \partial_{\bar{z}} \psi + \bar{\psi}\ \partial_z \bar{\psi}\ \mathrm{d}^2 x, \tag{87}$$

and perturb it with a mass term $-im\bar{\psi}\psi$. From this once can calculate $\langle \Theta(x)\Theta(0)\rangle$, which leads to

$$\langle \Theta(x)\Theta(0)\rangle = \left(\frac{m^2}{2\pi}\right)^2 \left[K_1^2(mr) - K_0^2(mr)\right] \tag{88}$$

where K_0, K_1 are Bessel functions. Plugging this back into eq. (86) we find together with $c_1 = 1/2$

$$\Delta c = 1/2 \qquad \Rightarrow \qquad c_2 = 0 \tag{89}$$

which is the central charge of a purely massive field theory in 2 dimensions [4]. We thus see that this mass perturbation generates an RG flow to a theory which is purely massive theory and therefore have checked explicitly that $c = 0$ for a massive theory [4]. An application of the c-theorem is thus to explicitly calculate the values of c for another theory.

10 Cardy's a-theorem

We would like to see if there is a c-theorem in 4 dimensions. We can first try Zamolochikov's approach; define correlation functions as eq. (31) -eq. (36) which are proportional to the correlation function of the stress energy tensor with itself. And from these definitions try to define a c-function of which the derivative is proportional to : $< \Theta(x)\Theta(0) >$ where $\Theta = T^{\mu}_{\mu}$, such that all wanted properties listed in section 2 are satisfied.

10.1 Zamolodchikov's approach

The first step is to define a correlation function in d dimensions, which is invariant under parity transformations and which has conformal weight $2d$, where d is the number of dimensions. Invariance under parity tells us that we can only have an even number of terms proportional to r_λ, with $A(r)$ some parity invariant function

$$\begin{aligned} F_{\mu_1\mu_2...\mu_n} =&\ A\ r_{\mu_1}r_{\mu_2}....r_{\mu_n} \\ r_{\mu_k} &\to -r_{\mu_k} \\ F_{\mu_1\mu_2...\mu_n} &\to (-1)^n\ F_{\mu_1\mu_2...\mu_n} \end{aligned}$$

We see that n must be even.The second constraint tells us the norm of every term appearing should be proportional to r^{-2d} and thus that for every vector r_λ we add we should also divide by $|r|$ to have the correct conformal weight of $2d$. Moreover the correlator needs to be symmetric in its first two and last two indices, since the energy momentum tensor is symmetric in its indices. This leads to the following two point function:

$$\begin{aligned} < T_{\mu\nu}T_{\lambda\rho} >= (A/r^{2d+4})r_\mu r_\nu r_\lambda r_\rho \\ +(B/r^{2d+2})(r_\mu r_\nu \delta_{\lambda\rho} + r_\lambda r_\rho \delta_{\mu\nu}) \\ +(C/r^{2d+2})(r_\mu r_\lambda \delta_{\nu\rho} + r_\nu r_\lambda \delta_{\mu\rho} + r_\mu r_\rho \delta_{\nu\lambda} + r_\nu r_\rho \delta_{\mu\lambda}) \\ +(D/r^{2d})\delta_{\mu\nu}\delta_{\lambda\rho} + (E/r^{2d})(\delta_{\mu\lambda}\delta_{\nu\rho} + \delta_{\nu\lambda}\delta_{\mu\rho}) \end{aligned} \tag{90}$$

where A, B, C, D and E are invariant amplitudes dependent on position. We can define our c-function to be

$$c = -\frac{4}{d-1}[A + \frac{1}{2}(d^2+d+2)B + (d+3)C + \frac{1}{2}d(d+1)D + (d+1)E], \tag{91}$$

such that:

$$\dot{c} = -\frac{4}{d+1}\langle\Theta\Theta\rangle - 2(d-2)B. \tag{92}$$

We would like to stress that the derivation of equation (63) is not obvious. One must start from the conservation equation: $\partial^\mu T_{\mu\nu} = 0$ and the decomposition: $\Theta = \beta(g)\phi$. To get relations between the derivatives of these amplitudes $\dot{A}, \dot{B}, ...,$

just like we have shown is possible in the $d = 2$ case: see equations eq. (44) and eq. (49).

The main point is that Zamolodchikov his approach of proving the existence of a c-function by using correlators and invariant amplitudes, only works in $d = 2$ due to term proportional to $(d - 2)B$ in eq. (92). Because in $d \neq 2$. We have in a fixed point ($\Theta = 0$): $\dot{c} = 2(d - 2)B$, thus c is not stationary in a fixed point. Which is the second property we want our c-function to have (section 2). So we come to the conclusion that to show the existence of a c-function in 4 dimensions we need another approach.

11 Cardy's approach: A c-function in d dimensions

We know from the Weyl anomaly that when the theory is place on a curved background the expectation value of the trace of the stress energy momentum tensor is (up to normalization of $(2\pi)^{-1}$)[3]

$$\langle \Theta \rangle = -cR/12.$$

Now define the following c-function in $d = 2$:

$$c = -\frac{3}{2\pi} \int_{S^2} \langle \Theta \rangle \sqrt{g} \mathrm{d}^2 x.$$

On the 2-sphere with radius ρ, we have $R = 2/\rho^2$, such that we get

$$\langle \Theta \rangle = -\frac{\bar{c}}{12} \frac{2}{\rho^2} = -\frac{\bar{c}}{6\rho^2} \tag{93}$$

$$\rightarrow$$

$$c = -\frac{3}{2\pi} \int_{S^2} -\frac{\bar{c}}{6\rho^2} \sqrt{g} \mathrm{d}^2 x \tag{94}$$

$$c = +\frac{\bar{c}}{4\pi} \int \frac{1}{\rho^2} \rho^2 \sin\theta \, \mathrm{d}\theta \mathrm{d}\phi \tag{95}$$

$$c = +\frac{\bar{c}}{4\pi} 4\pi \tag{96}$$

$$c = \bar{c}. \tag{97}$$

We see that this definition of the c-function appears to be correct when the theory is placed on the 2-sphere. This idea can be generalized to d dimensions where d is an even number. The definition below of the c-function vanishes when d is odd:

$$c = (-1)^{d/2} a_d \int_{S^d} \langle \Theta \rangle \sqrt{g} \mathrm{d}^2 x \tag{98}$$

In conclusion, we have found a way of defining c that directly satisfies property 3 in section 2 [3].

11.1 Perturbing the action

As we saw before in section 9 we can perturb the action with an operator that generates an RG flow. In this case we can consider perturbing the action by some weakly relevant operator ϕ_0

$$S = S^* - \lambda_0 \int \phi_0 \sqrt{g} \mathrm{d}^d x \tag{99}$$

Where we want to renormalize this field: $\phi(x) = Z^{-1/2}\phi_0$. Where ϕ_0 has scaling dimension $d - y$ with $0 < y \ll 1$. Zamolodchikov showed that the following relation holds

$$\Theta = S(d)\beta(g)\phi(x) = S(d)\beta(g)Z^{-1/2}\phi_0(x) \tag{100}$$

where $S(d) = 2\pi^{d/2}(\Gamma(d/2))^{-1}$ which is just the inverse of the angular integral of the d-sphere: $\int \Omega_d$. So in order to calculate c we need to know: $< \phi_0 >, Z, \beta(g)$. We know that

$$\beta(g) = -yg - \frac{1}{2}S(d)bg^2 + O(g^3) \tag{101}$$

$$Z = 1 + 2S(d)bg/y + O(g^2) \tag{102}$$

$$\langle \phi_0 \rangle = \lambda_0 \int \langle \phi_0(0)\phi_0(r) \rangle \, \mathrm{d}^d r$$

$$+\frac{1}{2}\lambda_0^2 \int \langle \phi_0(0)\phi_0(r_1)\phi_0(r_2) \rangle \, \mathrm{d}^d r_1 \mathrm{d}^d r_2 \tag{103}$$

$$= \lambda_0 (2\rho)^{-d+2y} I_2 + \frac{1}{2}\lambda_0^2 (2\rho)^{-d+3y} I_3 \tag{104}$$

where I_2, I_3 are the integrals stated below:

$$I_2 = \pi^{d/2} \, \frac{\Gamma(-d/2 + y)}{\Gamma(y)} \tag{105}$$

$$I_3 = \pi^d \, \frac{\Gamma(y/2)^3 \Gamma(-d/2 + 3y/2)}{\Gamma(y)^3 \Gamma(d/2)} \tag{106}$$

The 2- and 3-point functions in equation eq. (103) are defined as follows: $\langle \phi_0(0)\phi_0(r) \rangle = r^{-2h}$ and $\langle \phi_0(0)\phi_0(r_1)\phi_0(r_2) \rangle = r_1^h r_2^h |r_1 - r_2|^h$ [5], where $h = d - y$ [3] the scaling dimension of the bare fields ϕ_0. Plugging in these definitions into eq. (103) should give (13) and (14) from Cardy his paper [3]. However we can not figure out where the factor of : $\frac{1}{(1+r_i^2)^y}$ comes from. Plugging in everything and integrating over the d-sphere gives

$$c(g) \propto -\frac{1}{2}yg^2 - \frac{1}{6}S(d)bg^3 = \tag{107}$$

$$-\frac{1}{2}g^2[y + \frac{1}{3}S(d)bg] \tag{108}$$

$$\dot{c}(g) \propto -yg - 1/2S(d)bg^2 = \beta(g) \tag{109}$$

So we see that $\dot{c}(g)$ is stationary when $\beta(g^*) = 0.$, directly satisfying property 2 from section 2. However we see that $\dot{c}$ is proportional to the β function. This β function is not a monotonically decreasing function so property 1 is not satisfied and we can not prove the existence of a c-function in four dimensions.

12 The possible existence of an a-Theorem

In the previous section we saw that it was not possible to have a c-theorem in d dimensions, since the monotonic decreasing property was not proven. We would like to know if there can be an c-theorem in 4 dimensions (henceforth called a-theorem) based on the knowledge we have by looking at a theory where we know the degrees of freedom (the values of $c(g)$) in the IR and in the UV. We want to do this for QCD. We know that in the UV the coupling g goes to zero. The coupling g becomes infinitely strong in the IR. We know the degrees of freedom (the value of the c-function in these limits) to be as follows, where N_f are the amount of massless Dirac fermions and N_c are the gauge bosons of the group $SU(N_c)$[3]

$$\lim_{\mathrm{g}\to 0} \mathrm{c(g)} = a_{UV} = 11N_cN_f + 62(N_c^2 - 1) \tag{110}$$

$$\lim_{\mathrm{g}\to \infty} \mathrm{c(g)} = a_{IR} = N_f^2 - 1. \tag{111}$$

The a-theorem, would be violated if

$$a_{UV} < a_{IR} \tag{112}$$

$$11N_cN_f + 62(N_c^2 - 1) < N_f^2 - 1 \tag{113}$$

$$N_f^2 - 11N_cN_f - 62(N_c^2 - 1) - 1 > 0 \tag{114}$$

Changing $> \to =$ and solving for N_f

$$N_f^{\pm} = \frac{11}{2}N_c \pm \frac{1}{2}\sqrt{121N_c^2 + 4[62(N_c^2 - 1) + 1]} \tag{115}$$

$$N_f^{\pm} = \frac{11}{2}N_c \pm \sqrt{\left(\frac{11N_c}{2}\right)^2 + 62(N_c^2 - 1) + 1} \tag{116}$$

Sketching a_{UV} and a_{IR} fig. 1 as a function of N_f and only looking only at the positive solution we conclude that the a - theorem is not true if

$$N_f^{+} > \frac{11}{2}N_c + \sqrt{\left(\frac{11N_c}{2}\right)^2 + 62(N_c^2 - 1) + 1} \tag{117}$$

However a_{UV} is valid up to $N_f = \frac{11N_c}{2}$ so we (fig. 1) that a_{UV} is always bigger than a_{IR} so the existence of an a-function could be proven.

13 Current status and future perspectives

So Cardy could not prove the a -theorem, but he conjectured the existence of it [3]. In a paper by Osborn in 1989 [6], Osborn showed that an a-theorem exists when using pertubation theory on a curved space. In a paper by Komargodski & Schwimmer [7] it was shown that the a-theorem is also valid in a non-perturbative approach.

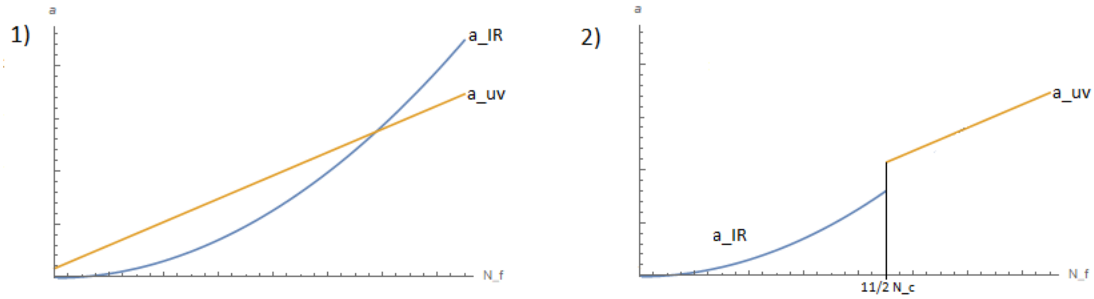

Figure 1: Sketch of the degrees of freedom (y-axis), in the IR and in the UV of QCD, as a function of the number of massless Dirac fermions N_f,. Plot 1) displays the degrees of freedom (d.o.f.) before the cutoff is imposed. Plot 2) displays the d.o.f. after the cutoff is imposed. The cutoff is at $N_f = \frac{11 N_c}{2}$, beyond this value the theory becomes asymptotically free (thus we are in the UV). Below this value the theory is strongly bound and we are in the IR.

14 Literature

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