Integrability & the Bethe Ansatz

The Lieb–Liniger model

Bosons in 1d $\quad [\psi(x), \psi^\dagger(x')] = \delta(x-x')$

$$H^{(N)}_{LL} = \sum_{j=1}^{N} -\frac{\partial^2}{\partial x_j^2} + 2c \sum_{j_1<j_2} \delta(x_{j_1}-x_{j_2})$$

Equiv:
$$H_{LL} = \int dx\, \mathcal{H}(x)$$

$$\mathcal{H}(x) = \partial_x \psi^\dagger(x)\, \partial_x \psi(x) + c\, \psi^\dagger(x)\psi^\dagger(x)\psi(x)\psi(x)$$

Build our states from Fock vacuum

$|0\rangle$ s.t. $\psi(x)|0\rangle = 0$. $\quad \langle 0|0\rangle = 1$

States in N-particle sector:

$$|\Psi_N\rangle = \int dx_1 \ldots dx_N\, \Psi_N(x_1,\ldots,x_N)\, \psi^\dagger(x_1)\ldots\psi^\dagger(x_N)|0\rangle$$

Schrödinger: $\quad H_{LL}|\Psi_N\rangle = E_N|\Psi_N\rangle$

Equiv.: $\quad H^{(N)}_{LL}\Psi_N(\underset{\sim}{x}) = E_N \Psi_N(\underset{\sim}{x})$

2-particle sector.

$$H = -\partial_{x_1}^2 - \partial_{x_2}^2 + 2c\,\delta(x_1 - x_2)$$

Idea: integrate in rel. coordinate $x_1 - x_2$ between $-\varepsilon$ & $+\varepsilon$ $\quad \varepsilon \to 0^+$

→ get a "boundary cond$^{\underline{n}}$":

$$\left(\partial_{x_2} - \partial_{x_1} - c\right) \Psi_2(x_1, x_2)\Big|_{x_2 - x_1 = 0^+} = 0 \qquad (32)$$

Away from $x_1 = x_2$: sol$^{\underline{n}}$ is a product of plane waves, e.g.

$$e^{i\lambda_1 x_1 + i\lambda_2 x_2} \qquad e^{i\lambda_2 x_1 + i\lambda_1 x_2}$$

$$E = \lambda_1^2 + \lambda_2^2 \qquad P = \lambda_1 + \lambda_2$$

Two-particle wavef$^{\underline{n}}$:

$$\Psi_2(x_1, x_2 | \lambda_1, \lambda_2) = S_1 e^{i\lambda_1 x_1 + i\lambda_2 x_2} + S_2 e^{i\lambda_2 x_1 + i\lambda_1 x_2}$$

Plug this into (32), get

$$\frac{S_2}{S_1} = -\frac{c + i(\lambda_1 - \lambda_2)}{c - i(\lambda_1 - \lambda_2)} = -e^{i\phi(\lambda_1 - \lambda_2)}$$

$$\phi(\lambda) \equiv \frac{1}{i} \ln \frac{c + i\lambda}{c - i\lambda} = 2\,\mathrm{atan}\left(\frac{\lambda}{c}\right)$$

Exact form of 2-particle wavef^{n}:

$$\psi_2(x_1, x_2|\lambda_1,\lambda_2)\Big|_{x_1<x_2} = e^{i\lambda_1 x_1 + i\lambda_2 x_2 - \frac{i}{2}\phi(\lambda_1-\lambda_2)}$$
$$- e^{i\lambda_2 x_1 + i\lambda_1 x_2 + \frac{i}{2}\phi(\lambda_1-\lambda_2)}$$

N-body wavef^{n}: simple generaliz^{n}:

$$x_1 < x_2 \ldots < x_N$$

BCs:
$$\left(\partial_{x_{j+1}} - \partial_{x_j} - c\right)\psi_N(x_1 \ldots x_N)\Big|_{x_{j+1}-x_j=0^+} = 0$$

N-body wavef^{n}: Bethe Ansatz

$$\boxed{\psi_N(\underset{\sim}{x}|\underset{\sim}{\lambda}) = (\mathrm{sgn}) \sum_{P\in\Pi_N} (-1)^P \times e^{i\sum_{j=1}^{N}\lambda_{P_j}x_j + \frac{i}{2}\sum_{N\geq j_1>j_2\geq 1} \mathrm{sgn}(x_{j_1}-x_{j_2})\phi(\lambda_{P_{j_1}}-\lambda_{P_{j_2}})}}$$

Quantization & Bethe equations

Impose PBC: system on circle of circumference L

Impose: $\Psi_2(x_2, x_1+L\,|\,\lambda_1,\lambda_2) = \Psi_2(x_1,x_2\,|\,\lambda_1,\lambda_2)$

By direct subst. of BA: must have

$$e^{i\lambda_1 L} = -e^{-i\phi(\lambda_1-\lambda_2)} \quad \& \quad e^{i\lambda_2 L} = -e^{i\phi(\lambda_1-\lambda_2)}$$

λ_2 

2 coupled nonlinear eq^{ns} for λ_1, λ_2.

Equiv: $$e^{i\lambda_1 L} = \frac{\lambda_1-\lambda_2+ic}{\lambda_1-\lambda_2-ic} \qquad e^{i\lambda_2 L} = \frac{\lambda_2-\lambda_1+ic}{\lambda_2-\lambda_1-ic}$$

<u>Bethe eq^{ns}</u>

More useful form: take logs

$$\lambda_1 + \frac{1}{L}\phi(\lambda_1-\lambda_2) = \frac{2\pi}{L} I_1$$

$$\lambda_2 + \frac{1}{L}\phi(\lambda_2-\lambda_1) = \frac{2\pi}{L} I_2$$

$I_j \in \mathbb{Z} + \frac{1}{2}$

Energy: $\lambda_1^2 + \lambda_2^2$

Mom: $\lambda_1 + \lambda_2$

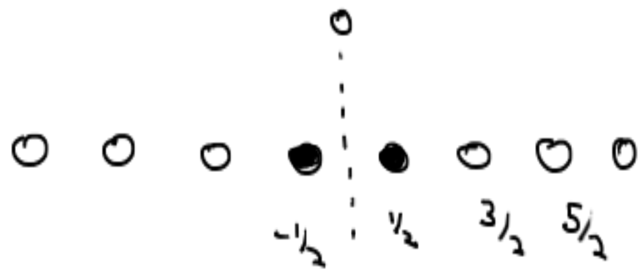
For N particles: PBCs $\Rightarrow$ Bethe eq$^{\underline{ns}}$

$$e^{i\lambda_j L} = \prod_{k\neq j} \frac{\lambda_j-\lambda_k+ic}{\lambda_j-\lambda_k-ic} \qquad j=1,\dots,N$$

log form:
$$\lambda_j + \frac{1}{L}\sum_{l=1}^{N}\phi(\lambda_j-\lambda_l) = \frac{2\pi}{L} I_j$$
$$j=1,\dots,N$$

$$I_j \in \begin{cases} \mathbb{Z}+\frac{1}{2} & N \text{ even} \\ \mathbb{Z} & N \text{ odd} \end{cases}$$

$$E = \sum_{j=1}^{N} \lambda_j^2 \qquad P = \sum_{j=1}^{N} \lambda_j$$



Properties of sol$^{\underline{ns}}$ to BE: $(c>0)$

1° all sol$^{\underline{ns}}$ are <u>real</u>

2° $\forall$ proper* sets $\{I_j\}$ $\{\lambda\}$ is unique

*: $I_j \neq I_k \ \forall j\neq k$

3° λ's are ordered as the I's.

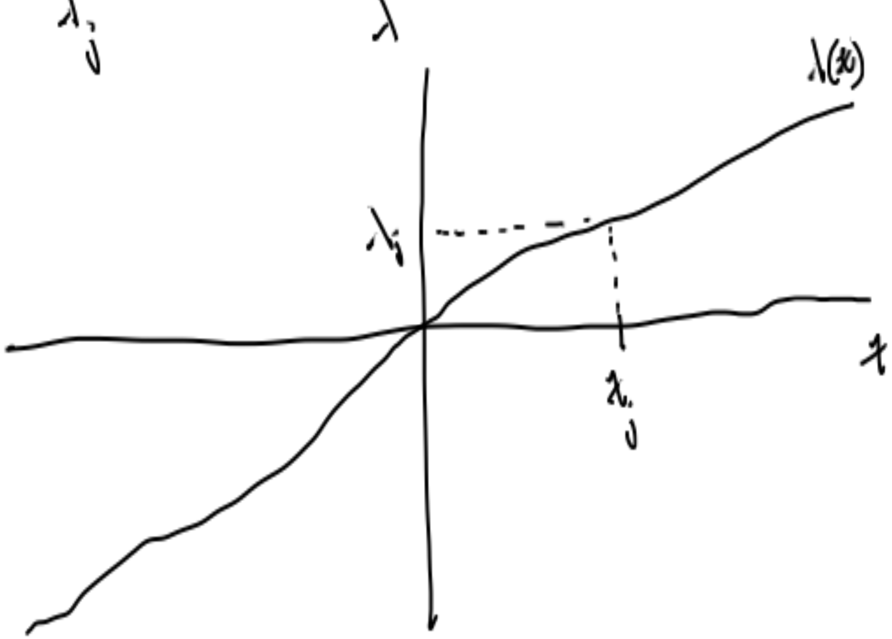
4° $$\frac{\frac{2\pi}{L}|I_j - I_l|}{1+\frac{2}{c}\frac{N}{L}} \leq |\lambda_j-\lambda_l| \leq \frac{2\pi}{L}|I_j-I_l|$$

Th. Lim: $N \to \infty$, $L \to \infty$, $\frac{N}{L} \to$ cst

Quantum #: $\frac{I_j}{L} \equiv x_j$

"Continuum" v$^{\underline{n}}$ of BE: $j = 1, \ldots, N$

$$\underset{\lambda_j}{\underset{\shortparallel}{\lambda(x_j)}} + \frac{1}{L} \sum_{l=1}^{N} \phi\left(\lambda(x_j) - \lambda(x_l)\right) = 2\pi x_j$$



Define a "density of occupation" of quantum # in x-space:

$$\rho(x) = \frac{1}{L} \sum_{j=1}^{N} \delta\left(x - \frac{I_j}{L}\right)$$

so

$$\lambda(x) + \int_{-\infty}^{\infty} dy\, \phi\left(\lambda(x) - \lambda(y)\right) \rho(y) = 2\pi x$$

We also have a "density of holes" ρ_h

$$\rho_h(x) = \frac{1}{L} \sum_{I \notin \{I_j\}} \delta\left(x - \frac{I}{L}\right)$$

s.t. $\underbrace{\rho(x) + \rho_h(x)}_{= \rho_t(x)} = 1$ as $L \to \infty$

"Move" to λ-space:

$$\rho(\lambda) = \rho(x(\lambda))\frac{dx(\lambda)}{d\lambda}$$

& same for $\rho_h(\lambda)$ $\qquad \rho_z(\lambda) = \frac{dx(\lambda)}{d\lambda}$

$$\lambda + \int_{-\infty}^{\infty} d\lambda' \, \phi(\lambda-\lambda')\rho(\lambda') = 2\pi x(\lambda)$$

Now: take $\frac{d}{d\lambda}$ of this:

$$1 + 2\pi\int_{-\infty}^{\infty} d\lambda' \, \mathcal{C}(\lambda-\lambda')\rho(\lambda') = 2\pi\left(\rho(\lambda)+\rho_h(\lambda)\right)$$

$$\mathcal{C}(\lambda) \equiv \frac{1}{2\pi}\frac{d}{d\lambda}\phi(\lambda) = \frac{1}{\pi}\frac{c}{\lambda^2+c^2}$$

Use $*$ not^{n}: $f*g(\lambda) \equiv \int_{-\infty}^{\infty} d\lambda' f(\lambda-\lambda')g(\lambda')$

$$\rightarrow \boxed{\rho(\lambda)+\rho_h(\lambda) = \frac{1}{2\pi} + \mathcal{C}*\rho(\lambda)} \quad (92)$$

BE

same:

$$\rho(\lambda) - \int_{-\infty}^{\infty} d\lambda' \, \mathcal{C}(\lambda-\lambda')\rho(\lambda') + \rho_h(\lambda) = \frac{1}{2\pi}$$

Bethe eq^{n}s in Toolbox.

Simplest case: ground state

Require: $\rho_h(\lambda)=0$ in interval $[-\lambda_F,\lambda_F]$

So

$$\rho_{GS}(\lambda)-\int_{-\lambda_F}^{\lambda_F}d\lambda'\,\mathcal{C}(\lambda-\lambda')\rho_{GS}(\lambda')=\frac{1}{2\pi}\quad \text{for } \lambda\in[-\lambda_F,\lambda_F]$$

$\rightarrow$ Lieb eq^n for the $\rho_{GS}(\lambda)$

representing the GS.

$$n=\int_{-\lambda_F}^{\lambda_F}d\lambda\,\rho_{GS}(\lambda)\qquad e=\int_{-\lambda_F}^{\lambda_F}d\lambda\,\lambda^2\rho_{GS}(\lambda)$$

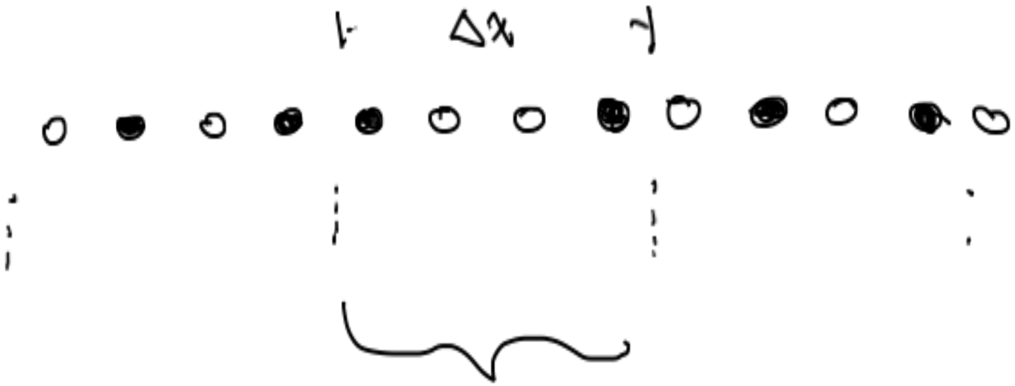
Equilibrium thermodynamics

(Yang-Yang formalism)

$$Z=\mathrm{Tr}\,e^{-(E-\mu N)/T}$$

Idea: write this as a funct^l integral over $\rho(\lambda)$.

$$Z=\sum_{N=0}^{\infty}\sum_{\{I\}_N}e^{-(E_{\{I\}}-\mu N)/T}$$



$$S_x = (\rho+\rho_h)\ln(\rho+\rho_h) - \rho\ln\rho - \rho_h\ln\rho_h$$

$$N_{boxes} = L\rho_t\Delta x \qquad \rho_t = \rho+\rho_h$$

$$N_{part} = L\rho\Delta x$$

$$N_{holes} = L\rho_h\Delta x$$

→ Yang-Yang entropy

$$S = L\int_{-\infty}^{\infty} dx\left[(\rho+\rho_h)\ln(\rho+\rho_h) - \rho\ln\rho - \rho_h\ln\rho_h\right]$$

# of equiv. config:

$$\binom{N_{boxes}}{N_{part}} = \frac{[L(\rho+\rho_h)\Delta x]!}{[L\rho\Delta x]!\,[L\rho_h\Delta x]!} \simeq e^{S_x \Delta x}$$

"entropy"

# of states "equivalent" to density ρ is

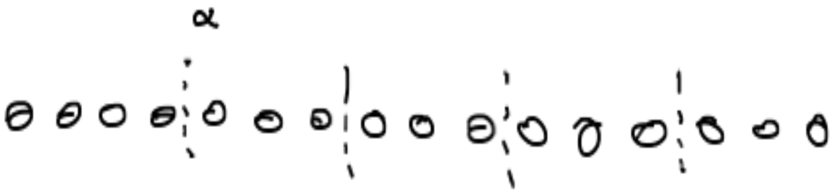
$$e^{S[\rho]}$$

For $\Delta x \to 0$ while $L\Delta x \to \infty$, approx with Stirling

$$\ln N! \simeq N\ln N - N + \cdots$$

Aim: compute $Z = \mathrm{Tr}\, e^{-\beta[H-\mu N]}$

$$\sum_{N=0}^{\infty} \sum_{\{I\}_N}$$



ρ_α = density of ● in box α

$\rho_{h,\alpha}$ = " ○ "

$\{\rho_\alpha\}\ \forall\alpha \longrightarrow \rho(x)$ fn

$$\prod_\alpha \int_0^1 d\rho_\alpha \longrightarrow \int \mathcal{D}\rho(x)$$

so

$$Z \longrightarrow \int \mathcal{D}\rho(x)\, e^{S[\rho]-\beta\{E[\rho]-\mu N\}}$$

At this pt: convenient to go from $\rho(x)$ to $\rho(\lambda)$

$$\prod_\alpha \int_0^1 d\rho_\alpha\, e^{S[\rho]} (\dots) \text{ "=" } \sum_N \sum_{\{I\}_N} (\dots)$$

$$\int \mathcal{D}\rho(x) \longrightarrow \int \mathcal{D}[\rho(\lambda), \rho_h(\lambda)] (\dots) \Big|_{BE}$$

In terms of $\rho(\lambda)$: $E[\rho] = L\int_{-\infty}^{\infty} d\lambda\, \lambda^2 \rho(\lambda)$

$N[\rho] = L\int_{-\infty}^{\infty} d\lambda\, \rho(\lambda)$

$$S[\rho, \rho_h] = L\int_{-\infty}^{\infty} d\lambda \left[(\rho+\rho_h)\ln(\rho+\rho_h) - \rho\ln\rho - \rho_h\ln\rho_h\right]$$

$$Z = \int \mathcal{D}[\rho,\rho_h]\, \underbrace{e^{S-\beta[E-\mu N]}}$$

$\rho + \rho_h = \frac{1}{2\pi} + \exp$

$e^{HO(1)}$ $\qquad$ $e^{-\beta F}$

$G = E - \mu N - TS$

Seek $\rho(\lambda)$ s.t. G is minim

Vary $\rho(\lambda) \to \rho(\lambda) + \delta\rho(\lambda)$

$\rho_h(\lambda) \to \rho_h + \delta\rho_h$

$$\frac{G}{L} = \int d\lambda \Big\{ \rho(\lambda)(\lambda^2 - \mu) - T(\rho+\rho_h)\ln(\rho+\rho_h) + T\rho\ln\rho + T\rho_h\ln\rho_h \Big\}$$

$$\frac{G[\rho+\delta\rho]}{L} - \frac{G[\rho]}{L} = \int d\lambda \Big\{ \delta\rho(\lambda)(\lambda^2-\mu) - T(\delta\rho+\delta\rho_h)\ln(\rho+\rho_h) - T\cancel{(\delta\rho+\delta\rho)}^2 + T\delta\rho\ln\rho + T\cancel{\delta\rho} + T\delta\rho_h\ln\rho_h + T\cancel{\delta\rho_h}^2 \Big\}$$

$$= \int d\lambda \Big\{ \delta\rho(\lambda)\Big[\lambda^2 - \mu - T\ln\Big(1+\frac{\rho_h}{\rho}\Big)\Big] - \delta\rho_h T\ln\Big(1+\frac{\rho}{\rho_h}\Big) \Big\}$$

Should remember: $\delta\rho$ & $\delta\rho_h$ are <u>not</u> indep

$\to$ linked by BE $\quad \rho + \rho_h = \frac{1}{2\pi} + \mathcal{C} * \rho$

so $\delta\rho_h = -\delta\rho + \mathcal{C} * \delta\rho$

Substituting back in,

$$\frac{\delta G}{L} = \int d\lambda\, \delta\rho(\lambda) \left\{ \lambda^2 - \mu - T\ln\left[1 + \frac{\rho_h}{\rho}\right] - \mathcal{C} * T\ln\left[1 + \rho/\rho_h\right] \right\}$$

Extremum: $\{ \} = 0$ $\quad$ Handy redef^n: $T\ln\frac{\rho_h}{\rho} \equiv \varepsilon(\lambda)$

$$\to \boxed{\varepsilon(\lambda) = \lambda^2 - \mu - \mathcal{C} * T\ln\left[1 + e^{-\varepsilon(\lambda)/T}\right]}$$

Yang-Yang eq^n

Free energy: rewritten as

$$\frac{G}{L} = -T\int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \ln\left[1 + e^{-\varepsilon(\lambda)/T}\right]$$

$$\mathcal{C}(\lambda) = \frac{1}{\pi}\frac{c}{\lambda^2 + c^2}$$

2 simple limits:

a) $c \to \infty \quad \mathcal{C}(\lambda) \to 0$

so $\varepsilon(\lambda) = \lambda^2 - \mu$

b) $c \to 0 \quad \mathcal{C}(\lambda) \to \delta(\lambda)$

so $\varepsilon(\lambda) = \lambda^2 - \mu - T\ln\left[1 + e^{-\varepsilon/T}\right]$

...

$$\to \frac{G}{L} = T\int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \ln\left[1 - e^{-(\lambda^2 - \mu)/T}\right]$$

Heisenberg spin chains

N sites

$S = 1/2$ degrees of freedom

$$H_{XXZ} = J \sum_{j=1}^{N} \left\{ \underbrace{S^x_j S^x_{j+1} + S^y_j S^y_{j+1}}_{= \frac{1}{2}(S^+S^- + S^-S^+)} + \Delta \left(S^z_j S^z_{j+1} - \frac{1}{4} \right) \right\}$$

Limits $\Delta \to \infty$: Ising

$\Delta \to 0$: XY model

Solvable with BA:

Idea: start from an "easy" eigenstate:

$$|0\rangle \equiv |\uparrow \dots \uparrow\rangle$$

(0 energy)

"add" $\downarrow$ spins $\to$ "particles"