Statistical Physics & Condensed Matter Theory I: Exercise solution

The Bohm-Staver formula: solutions

a)

For the free electron gas (including a factor 2 for spin)

$$\frac{N_e}{V} = 2\frac{1}{V}\sum_{\mathbf{k}}\theta(k_F - |\mathbf{k}|) = 2\int_{|\mathbf{k}|<k_F}\frac{d^3k}{(2\pi)^3} = 2\frac{4\pi}{(2\pi)^3}\int_0^{k_F} dk k^2 = \frac{1}{3\pi^2}k_F^3.$$

$$\frac{E_e^0}{V} = 2\frac{1}{V}\sum_{\mathbf{k}}\theta(k_F - |\mathbf{k}|)\frac{\hbar^2|\mathbf{k}|^2}{2m} = 2\frac{\hbar^2}{2m}\int_{|\mathbf{k}|<k_F}\frac{d^3k}{(2\pi)^3}|\mathbf{k}|^2 = \frac{\hbar^2}{2m}\frac{2\times 4\pi}{(2\pi)^3}\int_0^{k_F} dk k^4 = \frac{\hbar^2}{10\pi^2 m}k_F^5.$$

Since $k_F = (3\pi^2)^{1/3}\rho_e^{1/3}$ from above, we also get $\frac{E_e^0}{V} = \frac{\hbar^2}{10\pi^2 m}(3\pi^2)^{5/3}\rho_e^{5/3}$. Also, from the two equations above, $\frac{E_e^0}{N_e} = \frac{E_e^0}{V}\times\frac{V}{N_e} = \frac{3\hbar^2}{10m}k_F^2$. Using the definition $\varepsilon_F = \frac{\hbar^2 k_F^2}{2m}$ then also gives $\frac{E_e^0}{N_e} = \frac{3}{5}\varepsilon_F$.

b)

For the pressure $P = -\frac{\partial E_e^0}{\partial V}|_{N_e}$, we can start by writing (from above)

$$E_e^0 = \frac{\hbar^2}{10\pi^2 m}(3\pi^2)^{5/3}N_e^{5/3}V^{-2/3}$$

immediately giving

$$P = -\frac{\partial E_e^0}{\partial V}|_{N_e} = \frac{2}{3}\frac{E_e^0}{V} = \frac{2}{5}\varepsilon_F\rho_e$$

where in the last step we have used $\frac{E_e^0}{N_e} = \frac{3}{5}\varepsilon_F$ obtained above.

We can thus write

$$\dot{\boldsymbol{\pi}} = -\nabla P = -\frac{2}{5}\nabla(\varepsilon_F(\rho_e)\rho_e)$$

But $\varepsilon_F(\rho_e) = \frac{\hbar^2}{2m}(k_F(\rho_e))^2 = \frac{\hbar^2}{2m}(3\pi^2)^{2/3}\rho_e^{2/3}$ so $\varepsilon_F(\rho_e)\rho_e = \frac{\hbar^2}{2m}(3\pi^2)^{2/3}\rho_e^{5/3}$. Taking the gradient thus gives

$$\nabla(\varepsilon_F(\rho_e)\rho_e) = \frac{5}{3}\varepsilon_F(\rho_e)\nabla\rho_e \qquad \rightarrow \qquad \dot{\boldsymbol{\pi}} = -\frac{2}{3}\varepsilon_F\nabla\rho_e.$$

Since $\rho_e = \rho_e^0 + \delta\rho_e$, with ρ_e^0 constant, we can write $\dot{\boldsymbol{\pi}} = -\frac{2}{3}\varepsilon_F\nabla\delta\rho_e$. Taking the divergence of this yields a term of order $(\delta\rho_e)^2$ which we drop, plus (also using $\rho_e = Z\rho_{ion}$ or rather $\delta\rho_e = Z\delta\rho_{ion}$)

$$\nabla\cdot\dot{\boldsymbol{\pi}} = -\frac{2}{3}\varepsilon_F\nabla^2\delta\rho_e = -\frac{2}{3}Z\varepsilon_F\nabla^2\delta\rho_{ion}$$

which when used in the time-derivative of the continuity equation gives the required Harmonic equation

$$M\partial_t^2\delta\rho_{ion} - \frac{2}{3}Z\varepsilon_F\nabla^2\delta\rho_{ion} = 0.$$

c)

Let us look for a solution in the form of a wave propagating along (for definiteness, say) the x-axis, $\delta\rho_{ion}(\mathbf{r},t) = \phi(x - v_s t)$ in which v_s is some velocity to be determined. This solves the harmonic equation provided $v_s = \sqrt{\frac{2Z\varepsilon_F}{3M}}$. Using the fact that $\varepsilon_F = \frac{\hbar^2 k_F^2}{2m} = \frac{m}{2}v_F^2$ then yields $v_s = \sqrt{\frac{Zm}{3M}}v_F$.

For a typical solid, we can take $M \simeq Zm_{proton} \simeq Z \times 2000m \simeq 10^5 m$ as mentioned in the question. For Z about say 30 (thing *e.g.* of copper), we get $v_s \simeq 10^{-2} v_F \simeq 10^4 m/s$ which is in the ballpark of the speed of sound we're accustomed to for metals.