Statistical Physics & Condensed Matter Theory I: Exercise

A (classical) field theory for the Ising model

Consider a square lattice in d dimensions. On each lattice site lives an **Ising spin** $S_{\mathbf{a}}$, which is a *classical* spin variable taking only two possible values, $S_{\mathbf{a}} = \pm 1$. The vector $\mathbf{a}$ labels the lattice sites. The **Ising model** is defined by the classical Hamiltonian

$$H_I = -\sum_{\mathbf{a},\mathbf{b}} J_{\mathbf{ab}} S_{\mathbf{a}} S_{\mathbf{b}} - \sum_{\mathbf{a}} H_{\mathbf{a}} S_{\mathbf{a}},$$

where $H_{\mathbf{a}}$ is a magnetic field (which can vary from one position to another, hence the index), and the interaction coefficients $J_{\mathbf{ab}} \equiv J(|\mathbf{a} - \mathbf{b}|)$ fall off quickly with distance.

The classical partition function of the Ising model can be written as the sum over all possible Ising spin configurations,

$$\mathcal{Z}_I = \sum_{\{S_{\mathbf{a}}\}} e^{-\beta H_I} \equiv \sum_{\{S_{\mathbf{a}}\}} e^{\sum_{\mathbf{a},\mathbf{b}} S_{\mathbf{a}} K_{\mathbf{ab}} S_{\mathbf{b}} + \sum_{\mathbf{a}} h_{\mathbf{a}} S_{\mathbf{a}}}.$$

where we have defined $K_{\mathbf{ab}} \equiv \beta J_{\mathbf{ab}}$ and $h_{\mathbf{a}} \equiv \beta H_{\mathbf{a}}$.

a) Let's consider the noninteracting problem first: we set $K_{\mathbf{ab}} = 0$ everywhere. Show that the classical partition function then becomes

$$\mathcal{Z}_I|_{K_{\mathbf{ab}}=0} = \prod_{\mathbf{a}} 2\cosh h_{\mathbf{a}}.$$

b) What prevents us from solving this exactly[1] is the presence of the interaction term. We therefore will 'decouple' this interaction term using a Hubbard-Stratonovich transformation. Explicitly: starting from a representation of the unit operator as a field integral over an auxiliary (real, bosonic) field ψ,

$$\mathbf{1} \equiv \mathcal{N} \int \mathcal{D}\psi e^{-\frac{1}{4}\sum_{\mathbf{ab}} \psi_{\mathbf{a}} (K^{-1})_{\mathbf{ab}} \psi_{\mathbf{b}}}, \qquad \mathcal{D}\psi \equiv \prod_{\mathbf{a}} d\psi_{\mathbf{a}}$$

with K^{-1} the matrix inverse of K (and the normalization $\mathcal{N} = \left[\det\,(4\pi K^{-1})\right]^{1/2}$), show that the partition function of the interacting Ising theory can be written

$$\mathcal{Z}_I = \mathcal{N} \int \mathcal{D}\psi \sum_{\{S_{\mathbf{a}}\}} e^{-\frac{1}{4}\sum_{\mathbf{ab}} \psi_{\mathbf{a}} (K^{-1})_{\mathbf{ab}} \psi_{\mathbf{b}} + \sum_{\mathbf{a}} (\psi_{\mathbf{a}} + h_{\mathbf{a}}) S_{\mathbf{a}}}$$

[1] Note that the solution to the $d = 1$ Ising theory (in any field $h_{\mathbf{a}} = h$) is straightforward; in $d = 2$, Onsager offered the solution at zero field; the $d = 3$ case remains one of the most famous outstanding problems of classical statistical physics.

c) Perform the summation over all possible spin configurations ($\sum_{\{S_\mathbf{a}\}}$), and show that (after a redefinition of the field $\phi_\mathbf{a} \equiv \frac{1}{2}\sum_\mathbf{b}(K^{-1})_{\mathbf{ab}}(\psi_\mathbf{b}+h_\mathbf{b})$, inessential constant terms being reabsorbed into a redefinition of the constant $\mathcal{N} \to \tilde{\mathcal{N}}$) we get to the representation

$$\mathcal{Z}_I = \tilde{\mathcal{N}} \int \mathcal{D}\phi e^{-S[\phi]}, \qquad S[\phi] \equiv \sum_{\mathbf{ab}} \phi_\mathbf{a} K_{\mathbf{ab}} \phi_\mathbf{b} - \sum_\mathbf{a} \phi_\mathbf{a} h_\mathbf{a} - \sum_\mathbf{a} \ln\cosh(2\sum_\mathbf{b} K_{\mathbf{ab}}\phi_\mathbf{b})$$

d) We now make a low-temperature approximation, by assuming that the fluctuations of ϕ are small (for our purposes here, this translates into taking $|\phi_\mathbf{a}| \ll 1$). We define the Fourier representations (N is the total number of Ising spins in the system, and we assume periodic boundary conditions in all directions for simplicity) of the field and interaction as

$$\phi_\mathbf{a} = \frac{1}{\sqrt{N}} \sum_\mathbf{k} e^{-i\mathbf{k}\cdot\mathbf{a}} \phi_\mathbf{k}, \qquad K_{\mathbf{ab}} = \frac{1}{N}\sum_\mathbf{k} e^{-i\mathbf{k}\cdot(\mathbf{a}-\mathbf{b})} K_\mathbf{k},$$

and assume that the interaction is mostly local in space, so that we can use a small-momentum expansion (also called the gradient expansion) of the interaction, $K_\mathbf{k} = K_0 + \frac{1}{2}\mathbf{k}\cdot\mathbf{k}\, K_0^{''} + \mathrm{O}(|\mathbf{k}|^4)$. Using this and the expansion $\ln\cosh x = \frac{1}{2}x^2 - \frac{1}{12}x^4 + ...$, show that the action has the form

$$S[\phi] = \sum_\mathbf{k} \left[\phi_\mathbf{k}(c_1 + c_2 \mathbf{k}\cdot\mathbf{k})\phi_{-\mathbf{k}} + c_3 \phi_\mathbf{k} h_{-\mathbf{k}}\right] + \frac{c_4}{N} \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} \phi_{\mathbf{k}_1}\phi_{\mathbf{k}_2}\phi_{\mathbf{k}_3}\phi_{\mathbf{k}_4}\delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4,0} + ...$$

(obtain an expression for the coefficients c_i).

e) Upon Fourier transforming back to real space and a rescaling (you can put the field h to zero), the action becomes the so-called ϕ^4 theory,

$$\mathcal{Z} = \int \mathcal{D}\phi e^{-S[\phi]}, \qquad S[\phi] = \int d^d x \left[\frac{1}{2}(\partial\phi)^2 + \frac{r}{2}\phi^2 + g\phi^4\right].$$

Give the coefficients r and g in terms of the coefficients $J_0 \equiv \beta^{-1}K_0$ and $J_0^{''} \equiv \beta^{-1}K_0^{''}$. At what temperature does the Ising model have a transition ? *For your information: the Ising model has a transition when r changes sign from positive values to negative ones, i.e. when the potential becomes a 'Mexican hat' like potential. This is the field-theoretic way of describing this transition.*