Statistical Physics & Condensed Matter Theory I: Exercise

A (classical) field theory for the Ising model

a) Trivial since $\sum_{S_{\mathbf{a}}} e^{-\beta h_{\mathbf{a}} S_{\mathbf{a}}} = 2\cosh h_{\mathbf{a}}$.

b) We here consider the given representation of unity, but for a shifted value of the field:

$$\mathbf{1} \equiv \mathcal{N} \int \mathcal{D}\psi e^{-\frac{1}{4}\sum_{\mathbf{ab}}(\psi_{\mathbf{a}}+\theta_{\mathbf{a}})(K^{-1})_{\mathbf{ab}}(\psi_{\mathbf{b}}+\theta_{\mathbf{a}})}, \qquad \mathcal{D}\psi \equiv \prod_{\mathbf{a}} d\psi_{\mathbf{a}}$$

in which $\theta_{\mathbf{a}}$ are constants to be chosen (the integration measure doesn't change under such shifts by constants). Written out explicitly,

$$\mathbf{1} \equiv \mathcal{N} \int \mathcal{D}\psi e^{-\frac{1}{4}\sum_{\mathbf{ab}}\psi_{\mathbf{a}}(K^{-1})_{\mathbf{ab}}\psi_{\mathbf{b}} - \frac{1}{2}\sum_{\mathbf{ab}}\theta_{\mathbf{a}}(K^{-1})_{\mathbf{ab}}\psi_{\mathbf{b}} - \frac{1}{4}\sum_{\mathbf{ab}}\theta_{\mathbf{a}}(K^{-1})_{\mathbf{ab}}\theta_{\mathbf{b}}}$$
$$= e^{-\frac{1}{4}\sum_{\mathbf{ab}}\theta_{\mathbf{a}}(K^{-1})_{\mathbf{ab}}\theta_{\mathbf{b}}} \times \mathcal{N}\int \mathcal{D}\psi e^{-\frac{1}{4}\sum_{\mathbf{ab}}\psi_{\mathbf{a}}(K^{-1})_{\mathbf{ab}}\psi_{\mathbf{b}} - \frac{1}{2}\sum_{\mathbf{ab}}\theta_{a}(K^{-1})_{\mathbf{ab}}\psi_{\mathbf{b}}}$$

Choosing $\theta_{\mathbf{a}} = -2\sum_{\mathbf{b}} K_{\mathbf{ab}} S_{\mathbf{b}}$ gives

$$e^{\sum_{\mathbf{ab}} S_{\mathbf{a}} K_{\mathbf{ab}} S_{\mathbf{b}}} = \mathcal{N}\int \mathcal{D}\psi e^{-\frac{1}{4}\sum_{\mathbf{ab}}\psi_{\mathbf{a}}(K^{-1})_{\mathbf{ab}}\psi_{\mathbf{b}} + \sum_{\mathbf{a}} S_{\mathbf{a}}\psi_{\mathbf{a}}}$$

which gives the required answer by direct substitution.

c) Simple.

d) Consider

$$x_{\mathbf{a}} = 2\sum_{\mathbf{b}} K_{\mathbf{ab}}\phi_{\mathbf{b}} = \frac{2}{N^{3/2}}\sum_{\mathbf{k},\mathbf{k}'}\sum_{\mathbf{b}} e^{-i\mathbf{k}\cdot(\mathbf{a}-\mathbf{b})-i\mathbf{k}'\cdot\mathbf{b}} K_{\mathbf{k}}\phi_{\mathbf{k}'} = \frac{2}{\sqrt{N}}\sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{a}} K_{\mathbf{k}}\phi_{\mathbf{k}} \tag{1}$$

We can then write

$$\sum_{\mathbf{a}} \frac{x_{\mathbf{a}}^2}{2} = \frac{2}{N}\sum_{\mathbf{k}_1,\mathbf{k}_2} K_{\mathbf{k}_1}K_{\mathbf{k}_2}\phi_{\mathbf{k}_1}\phi_{\mathbf{k}_2}\sum_{\mathbf{a}} e^{-i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{a}} = 2\sum_{\mathbf{k}} K_{\mathbf{k}}K_{-\mathbf{k}}\phi_{\mathbf{k}}\phi_{-\mathbf{k}} \tag{2}$$

and

$$\sum_{\mathbf{a}} \frac{x_{\mathbf{a}}^4}{12} = \frac{4}{3N^2}\sum_{\mathbf{k}_1\ldots\mathbf{k}_4} K_{\mathbf{k}_1}K_{\mathbf{k}_2}K_{\mathbf{k}_3}K_{\mathbf{k}_4}\phi_{\mathbf{k}_1}\phi_{\mathbf{k}_2}\phi_{\mathbf{k}_3}\phi_{\mathbf{k}_4}\sum_{\mathbf{a}} e^{-i(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\cdot\mathbf{a}} \tag{3}$$

Doing the gradient expansion in each of these, and keeping only leading terms,

$$\sum_{\mathbf{a}} \frac{x_{\mathbf{a}}^2}{2} \simeq 2\sum_{\mathbf{k}} (K_0 + \frac{1}{2}|\mathbf{k}|^2 K_0'' + \ldots)^2 \phi_{\mathbf{k}}\phi_{-\mathbf{k}} \tag{4}$$

$$\sum_{\mathbf{a}} \frac{x_{\mathbf{a}}^4}{12} = \frac{4K_0^4}{3N} \sum_{\mathbf{k}_1 \ldots \mathbf{k}_4} \phi_{\mathbf{k}_1}\phi_{\mathbf{k}_2}\phi_{\mathbf{k}_3}\phi_{\mathbf{k}_4}\delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4,0} \tag{5}$$

Putting things together, we get the form for the action given in the question, with coefficients

$$c_1 = K_0 - 2K_0^2, \qquad c_2 = \frac{1}{2}K_0^{''} - 2K_0 K_0^{''}, \qquad c_3 = -1, \qquad c_4 = \frac{4}{3}K_0^4 \tag{6}$$

e) Back to real space (specializing to $h = 0$)

$$S[\phi] = \int d^d x \left[c_2(\partial\phi)^2 + c_1\phi^2 + c_4\phi^4\right] \tag{7}$$

Rescaling $\phi \to \frac{1}{\sqrt{2c_2}}\phi$, we get $r = \frac{c_1}{c_2}$ and $g = \frac{c_4}{4c_2^2}$.

The calculation only makes sense if $c_2 > 0$ so $K_0 < 1/4$. r changes sign when $c_1 = 0$ so $K_0 = 1/2$ and thus the critical temperature is given by the condition $\beta_c = \frac{1}{2J_0}$. For $\beta > \beta_c$, $r < 0$ and the system is in the ordered phase. For $\beta < \beta_c$, $r > 0$ and the system is in the disordered phase.