Statistical Physics & Condensed Matter Theory I: Exercise

Grassmann variables

We have seen in class the (multivariable) Gaussian integration identities for bosonic variables (real case on the left, complex case on the right):

$$\int d\mathbf{v} e^{-\frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v}} = (2\pi)^{N/2} \det \mathbf{A}^{-1/2}, \qquad \int d(\mathbf{w}^\dagger, \mathbf{w}) e^{-\mathbf{w}^\dagger \mathbf{B}\mathbf{w}} = \pi^N \det \mathbf{B}^{-1}$$

where (for the left part) $\mathbf{A}$ is a positive definite real symmetric N-dimensional matrix, $\mathbf{v}$ is an N-component real vector, $d\mathbf{v} \equiv \prod_i dv_i$, and (for the right part) $\mathbf{w}$ an N-dimensional complex vector, $d(\mathbf{w}^\dagger, \mathbf{w}) \equiv \prod_{i=1}^N d\Re w_i d\Im w_i$, and $\mathbf{B}$ a complex matrix with positive definite Hermitian part.

For the case of Grassmann variables, we have only the equivalent of the 'right' part, namely

$$???, \qquad \int d(\bar{\phi}, \phi) e^{-\bar{\phi}^T \mathbf{B} \phi} = \det \mathbf{B}$$

where $\bar{\phi}$ and ϕ are independent N-component vectors of Grassmann variables, the measure is $d(\bar{\phi}, \phi) \equiv \prod_{i=1}^N d\bar{\phi}_i d\phi_i$ and $\mathbf{B}$ can be an *arbitrary* complex matrix.

In this exercise, we complete the table by providing the missing ??? piece in the above equation. Consider thus the following multidimensional Grassmann integration

$$I_G \equiv \int d\boldsymbol{\phi} e^{-\frac{1}{2}\sum_{i,j=1}^N \phi_i \mathbf{A}_{ij} \phi_j}$$

(take for definiteness $\int d\boldsymbol{\phi} \equiv \int d\phi_N \int d\phi_{N-1} ... \int d\phi_1$), where $\mathbf{A}$ is taken to be an antisymmetric matrix (no points: do you see why ?).

a) Compute this explicitly for $N = 1, 2$ and 3.

b) Show that this vanishes for any matrix if N is odd.

c) Show that for N even, $I_G = (-1)^{N/2}$ Pf $\mathbf{A}$ where

$$\text{Pf } \mathbf{A} \equiv \frac{1}{2^{N/2}(N/2)!} \sum_{i_1, i_2, ..., i_N = 1}^{N} \epsilon_{i_1 i_2 i_3 ... i_N} \mathbf{A}_{i_1 i_2} \mathbf{A}_{i_3 i_4} ... \mathbf{A}_{i_{N-1} i_N}$$

(in which $\epsilon_{i_1 i_2 ...}$ is the completely antisymmetric tensor in all indices, with $\epsilon_{12...N} \equiv 1$) is called the **Pfaffian** of matrix $\mathbf{A}$.

For your information: the Pfaffian of an antisymmetric matrix is such that $(Pf\ \mathbf{A})^2 = \det \mathbf{A}$ (an identity you do **not** *have to show here !), so this does offer a parallel to the 'square root of determinant' of the bosonic case.*