Statistical Physics & Condensed Matter Theory I: Exercise

Heisenberg chain with next-nearest-neighbour coupling: Solution

a)

For $J_1 \gg J_2$, the classical ground state is simply given by a fully-aligned spin configuration. The quantum ground states are in this case simply the same as the classical ones. The direction of this alignment is not important, and there is thus a degeneracy corresponding to uniformly rotating all spins. The existence of this symmetry leads to the existence of one type of low-energy spin wave modes.

For $J_2 \gg J_1$, an interesting situation occurs: spins on odd lattice sites tend to order antiferromagnetically with one another, and so do spins on even lattice sites. These two orderings are *independent*, and there are thus two distinct symmetries: uniform rotations of odd/even lattice site spins. Note that the J_1 term does *not* lift the degeneracy (to first order) associated to rotating *e.g.* even-site spins for a given odd-site spin antiferromagnetic order.

b)

Holstein-Primakoff: keep only $S_j^- = \sqrt{2S}a_j^\dagger$, $S_j^+ = \sqrt{2S}a_j$:

$$\begin{aligned} S_j \cdot S_{j+1} &= \frac{1}{2}\left(S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+\right) + S_j^z S_{j+1}^z \\ &= S(a_j a_{j+1}^\dagger + a_j^\dagger a_{j+1}) + (S - a_j^\dagger a_j)(S - a_{j+1}^\dagger a_{j+1}) + \mathcal{O}(S^0) \\ &= S^2 - S(a_{j+1}^\dagger - a_j^\dagger)(a_{j+1} - a_j) + \mathcal{O}(S^0) \end{aligned}$$

The calculation is identical for $S_j \cdot S_{j+2}$, so the effective bosonic theory is

$$H = -NS^2(J_1 - J_2) + S\sum_j \left[J_1(a_{j+1}^\dagger - a_j^\dagger)(a_{j+1} - a_j) - J_2(a_{j+2}^\dagger - a_j^\dagger)(a_{j+2} - a_j)\right] + \mathcal{O}(S^0)$$

c)

By Fourier transformation, we have e.g. $\sum_j (a_{j+1}^\dagger - a_j^\dagger)(a_{j+1} - a_j) = \sum_k |e^{ik} - 1|^2 a_k^\dagger a_k$. Since $|e^{ik} - 1|^2 = 4\sin^2\frac{k}{2}$, we can write

$$\begin{aligned} H &= -NS^2(J_1 - J_2) + S\sum_k \omega_k a_k^\dagger a_k + \mathcal{O}(S^0), \\ \omega_k &= 4J_1 \sin^2\frac{k}{2} - 4J_2\sin^2 k. \end{aligned}$$

Using the identity $\sin 2a = 2\sin a \cos a$, the spin-wave dispersion relation can be rewritten

$$\omega_k = 4J_1 \sin^2\frac{k}{2}\left(1 - \frac{4J_2}{J_1}\cos^2\frac{k}{2}\right).$$

If $J_2 > J_1/4$, there exists a region in k around $k = 0$ for which $\omega_k < 0$. Therefore, it would be energetically advantageous to create as many of these negative-energy excitations as possible to minimize the total energy, so the system would be unstable. We can thus have some faith in this calculation only for $J_2 < J_1/4$.