Statistical Physics & Condensed Matter Theory I: Exercise

Hubbard-Stratonovich decoupling

In class, we have seen that the action for fermions in 3d with Coulomb interactions is

$$S[\bar{\psi}, \psi] = \sum_p \sum_\sigma \bar{\psi}_{p\sigma}(-i\omega_n + \frac{\mathbf{p}^2}{2m} - \mu)\psi_{p\sigma} + \frac{T}{2L^3} \sum_{pp'q} \sum_{\sigma\sigma'} \bar{\psi}_{p+q\sigma} \bar{\psi}_{p'-q\sigma'} V(\mathbf{q}) \psi_{p'\sigma'} \psi_{p\sigma}$$

with $V(\mathbf{q}) = \frac{4\pi e^2}{|\mathbf{q}|^2}$. The interaction term is difficult to handle: we can treat it in perturbation theory, or be more fancy and decouple it using a Hubbard-Stratonovich transformation.

Rewriting the interaction in terms of the density operator $\rho_q \equiv \sum_{p\sigma} \bar{\psi}_{p+q\sigma}\psi_{p\sigma}$, and considering the following representation for a simple Gaussian integration over a bosonic field variable ϕ,

$$\int \mathcal{D}\phi e^{-S_0[\phi]} \equiv 1, \qquad S_0[\phi] = \frac{1}{2} \sum_q \phi_q V^{-1}(\mathbf{q}) \phi_{-q}$$

(in the left equation, the value of the right-hand side defines the normalization of the measure $\mathcal{D}\phi$, which you don't need to worry about), show that the original interacting theory's partition function

$$\mathcal{Z} = \int \mathcal{D}(\bar{\psi}, \psi) e^{-S[\bar{\psi}, \psi]}$$

can be rewritten in terms of a theory of fields $\phi, \bar{\psi}, \psi$ coupled to one another,

$$\mathcal{Z} = \int \mathcal{D}\phi \int \mathcal{D}(\bar{\psi}, \psi) e^{-S[\phi, \bar{\psi}, \psi]}.$$

Give the explicit form of the effective action $S[\phi, \bar{\psi}, \psi]$. This is called **decoupling in the direct channel**.