Statistical Physics & Condensed Matter Theory I: Exercise

The Jaynes-Cummings model

One of the most important recent breakthroughs in physics has been the ability to isolate and manipulate *single* atoms[1], this being done either in harmonic traps or optical cavities. The central model in this field is called the **Jaynes-Cummings** model, and describes a two-level atom interacting with the quantized modes of a harmonic oscillator (see the figure). This problem uses second quantization to extract some interesting physics from this model.

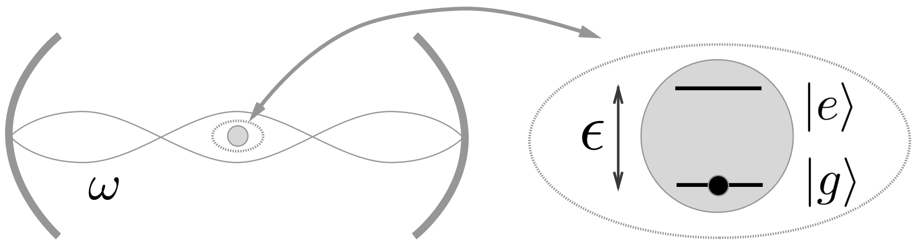

Figure 1: Example setup for the Jaynes-Cummings model. An optical cavity contains quantized modes of frequency ω. A two-level system (*e.g.* two internal states of an atom) with energy splitting ϵ is coupled to the cavity mode. The two-level system is here pictured in its ground state $|g\rangle$. Absorption of a quantum of the cavity mode promotes it to the excited state $|e\rangle$.

The Hamiltonian of the model is

$$H_{JC} = \frac{\epsilon}{2}\sigma^z + \omega a^\dagger a + \frac{\Omega}{2}(a\sigma^+ + a^\dagger \sigma^-).$$

The first term represents the two-level atom, the second represents the cavity mode (a quantum harmonic oscillator), and the third term is the coupling between these. We have represented the two-level atom as a pseudo-spin (denoting the ground state and excited state of the two-level system respectively as $|g\rangle$ and $|e\rangle$)

$$|g\rangle \equiv |\downarrow\rangle, \qquad |e\rangle \equiv |\uparrow\rangle,$$

with operators

$$\sigma^z = |e\rangle\langle e| - |g\rangle\langle g|, \quad \sigma^+ = |e\rangle\langle g|, \quad \sigma^- = |g\rangle\langle e|.$$

The Ω term means that the two-level atom can go from the ground to the excited state by absorbing one quantum of the cavity mode, and conversely relax from the excited to the ground state by emitting a quantum of the cavity mode. The set of states $|n, \uparrow (\downarrow)\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0, \uparrow (\downarrow)\rangle$, $n \in \mathbb{N}$ forms a basis for the Hilbert space.

[1]The 2012 Nobel Prize in Physics was awarded to Serge Haroche and David Wineland *'for ground-breaking experimental methods that enable measuring and manipulation of individual quantum systems'*.

a)

Show that the quantity

$$\hat{N} = a^\dagger a + \frac{1}{2}\sigma^z$$

commutes with the Jaynes-Cummings Hamiltonian, and is thus a conserved quantity.

b)

Let us now work in a fixed subspace of the Hilbert space in which the operator $\hat{N}$ takes on a definite value n. This subspace is two-dimensional, and we write its two basis states as

$$|n-1,\uparrow\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |n,\downarrow\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Show that when projected onto this subspace, the Hamiltonian is represented as

$$H_{JC}^{(n)} = \begin{pmatrix} \langle n-1,\uparrow|H_{JC}|n-1,\uparrow\rangle & \langle n-1,\uparrow|H_{JC}|n,\downarrow\rangle \\ \langle n,\downarrow|H_{JC}|n-1,\uparrow\rangle & \langle n,\downarrow|H_{JC}|n,\downarrow\rangle \end{pmatrix} = \omega(n-\frac{1}{2})\mathbf{1} + \frac{1}{2}\begin{pmatrix} \delta & \Omega\sqrt{n} \\ \Omega\sqrt{n} & -\delta \end{pmatrix},$$

the first term being proportional to the unit matrix (and thus representing a trivial constant energy shift for given n), and the parameter $\delta \equiv \epsilon - \omega$ being called the *detuning*.

c)

Diagonalize this using a Bogoliubov transformation (you can directly use the Useful Formulas below, without rederivation). Give the explicit form of the two eigenstates together with their energy.

d) Rabi oscillations

Consider now an initial state

$$|\psi(t=0)\rangle = |n,\downarrow\rangle.$$

Show that the propability $P_{exc}(t)$ of finding the system in the excited state $|n-1,\uparrow\rangle$ displays **Rabi oscillations** at frequency ω_R (to be determined) according to the formula

$$P_{exc}(t) = \frac{1}{2}(1-\cos\omega_R t)\frac{\Omega^2 n}{\Omega^2 n + \delta^2}.$$

e) Coherent initial state

Let us now consider a more interesting initial state where the bosons are in a *coherent state*:

$$|\psi(t=0)\rangle = \mathcal{N} e^{\lambda a^\dagger}|0,\downarrow\rangle, \qquad \lambda \in \mathbb{C}.$$

Setting the value of $\mathcal{N}$ so that the state is normalized, and considering the simplest case of zero detuning $\delta = 0$, show that the probability of finding the atom in the excited state (*i.e.* the two-level system in state $\uparrow$, irrespective of the number of quanta in the cavity mode) is given by

$$P_{exc}(t) = \frac{1}{2} - \frac{1}{2}e^{-|\lambda|^2}\sum_{n=0}^{\infty}\frac{|\lambda|^{2n}}{n!}\cos(\Omega\sqrt{n}t).$$

f)* Collapse and revival

Consider now the case $|\lambda| \gg 1$ (in other words, the cavity mode is very highly occupied). By using Stirling's formula $\ln(n!) = n \ln n - n + \ln \sqrt{2\pi n} + O(1/n)$ (for $n \gg 1$), argue that the summand is sharply peaked around a value $n_p \simeq |\lambda|^2$, and that by expanding around this peak value we can write the approximate result

$$P_{exc}(t) \simeq \frac{1}{2} - \frac{1}{2\sqrt{2\pi|\lambda|^2}} \operatorname{Re} \left(\sum_{m=-\infty}^{\infty} e^{-\frac{m^2}{2|\lambda|^2} + i\Omega t\sqrt{|\lambda|^2+m}} \right).$$

By looking at contributions around the peak at $m = 0$, looking at the set of 'fast' oscillations at frequency $\Omega|\lambda|$, and considering that the most important contributions come from values of m such that $|m| < |\lambda|$ (because others are suppressed by the Gaussian form), argue that there are three relevant time scales to the problem: a time scale for *oscillations*, one for *decay*, and one for *revival*, and that these are respectively given by

$$T_{osc} \simeq \frac{1}{\Omega|\lambda|}, \qquad T_{dec} \simeq \frac{1}{\Omega}, \qquad T_{rev} \simeq \frac{|\lambda|}{\Omega}.$$

For your information: the revival is a purely quantum effect (it would not exist if the cavity modes were not quantized), and is observable experimentally in one-atom masers.

Useful formulas

Bogoliubov transformation (bosonic case)

The matrix

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

(here for $a, b \in \mathbb{R}$) can be diagonalized by the unitary transformation

$$UHU^\dagger = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix}, \qquad U = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

where $\tan 2\theta = \frac{b}{a}$ and $\varepsilon = (a^2 + b^2)^{1/2}$.