Statistical Physics & Condensed Matter Theory I: Exercise

The Jaynes-Cummings model: Solution

a)

The only nontrivial commutators are

$$[H_{JC}, \hat{N}] = \frac{\Omega}{2}[a\sigma^+ + a^\dagger\sigma^-, a^\dagger a + \frac{1}{2}\sigma^z].$$

Explicitly,

$$[a\sigma^+, a^\dagger a] = [a, a^\dagger]a\sigma^+ = a\sigma^+, \qquad [a^\dagger\sigma^-, a^\dagger a] = a^\dagger[a^\dagger, a]\sigma^- = -a^\dagger\sigma^-,$$

$$[a\sigma^+, \sigma^z] = -2a\sigma^+, \qquad [a^\dagger\sigma^-, \sigma^z] = 2a^\dagger\sigma^-.$$

Since these all add to zero, we indeed have $[H_{JC}, \hat{N}] = 0$.

b)

Explicitly, we have

$$\sigma^z|n,\sigma\rangle = \sigma|n,\sigma\rangle, \qquad a^\dagger a|n,\sigma\rangle = n|n,\sigma\rangle, \qquad \sigma = \pm 1(\uparrow,\downarrow).$$

Since $\sigma^+|n,\uparrow\rangle = 0 = \sigma^-|n,\downarrow\rangle$, the only other nonzero matrix elements are

$$a\sigma^+|n,\downarrow\rangle = \sqrt{n}|n-1,\uparrow\rangle, \qquad a^\dagger\sigma^-|n-1,\uparrow\rangle = \sqrt{n}|n,\downarrow\rangle.$$

The action of the Hamiltonian on the two basis states is thus straightforwardly written as

$$H|n-1,\uparrow\rangle = [\frac{\epsilon}{2} + \omega(n-1)]|n-1,\uparrow\rangle + \frac{\Omega\sqrt{n}}{2}|n,\downarrow\rangle,$$
$$H|n,\downarrow\rangle = [-\frac{\epsilon}{2} + \omega n]|n,\downarrow\rangle + \frac{\Omega\sqrt{n}}{2}|n-1,\uparrow\rangle,$$

giving

$$H_{JC}^{(n)} = \begin{pmatrix} \frac{\epsilon}{2} + \omega(n-1) & \frac{\Omega}{2}\sqrt{n} \\ \frac{\Omega}{2}\sqrt{n} & -\frac{\epsilon}{2} + \omega n \end{pmatrix}$$

which gives the required form when substituting for the detuning.

c)

The Bogoliubov form is here

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \qquad \tan 2\theta = \frac{\Omega\sqrt{n}}{\delta}, \qquad U H_{JC}^{(n)} U^\dagger = \frac{1}{2}(\Omega^2 n + \delta^2)^{1/2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The two eigenstates are

$$U\begin{pmatrix}1\\0\end{pmatrix} = \cos\theta|n-1,\uparrow\rangle + \sin\theta|n,\downarrow\rangle \equiv |+\rangle,$$
$$U\begin{pmatrix}0\\1\end{pmatrix} = \sin\theta|n-1,\uparrow\rangle - \cos\theta|n,\downarrow\rangle \equiv |-\rangle,$$

their energies being

$$H_{JC}^{(n)}|\pm\rangle = \left[\omega(n-\frac{1}{2}) \pm \frac{1}{2}(\Omega^2 n + \delta^2)^{1/2}\right]|\pm\rangle \equiv E_\pm|\pm\rangle.$$

d) Rabi oscillations

By inspection, knowing that $\sin^2\theta + \cos^2\theta = 1$, we see that the initial state can be written as

$$|\psi(t=0)\rangle = |n,\downarrow\rangle = \sin\theta|+\rangle - \cos\theta|-\rangle.$$

The time-dependent wavefunction is then

$$|\psi(t)\rangle = e^{-iH_{JC}^{(n)}t}|\psi(t=0)\rangle = e^{-iE_+t}\sin\theta|+\rangle - e^{-iE_-t}\cos\theta|-\rangle.$$

The propability $P_{exc}(t)$ of finding the system in the excited state $|n-1,\uparrow\rangle$ is thus

$$P_{exc}(t) = |\langle n-1,\uparrow|\psi(t)\rangle|^2.$$

This can be calculated using the fact that $|n-1,\uparrow\rangle = \cos\theta|+\rangle + \sin\theta|-\rangle$ and that $|\pm\rangle$ are by construction orthonormal:

$$\langle n-1,\uparrow|\psi(t)\rangle = (\cos\theta\langle+| + \sin\theta\langle-|)(e^{-iE_+t}\sin\theta|+\rangle - e^{-iE_-t}\cos\theta|-\rangle)$$
$$= \cos\theta\sin\theta(e^{-iE_+t} - e^{-iE_-t}) = -i\sin 2\theta e^{-i\omega(n-1/2)t}\sin[\frac{1}{2}(\Omega^2 n+\delta^2)^{1/2}t].$$

We thus get

$$P_{exc}(t) = \sin^2 2\theta\sin^2[\frac{1}{2}(\Omega^2 n+\delta^2)^{1/2}t]$$

Using

$$\sin^2 2\theta = \tan^2 2\theta * \cos^2 2\theta = \frac{\tan^2 2\theta}{1+\tan^2 2\theta} = \frac{\Omega^2 n}{\Omega^2 n+\delta^2}$$

and the identity $\sin^2 2\theta = \frac{1}{2}(1-\cos 2\theta)$ gives the required form

$$P_{exc}(t) = \frac{1}{2}(1-\cos\omega_R t)\frac{\Omega^2 n}{\Omega^2 n+\delta^2}$$

displaying Rabi oscillations at frequency

$$\omega_R = (\Omega^2 n+\delta^2)^{1/2}.$$

e) Coherent initial state

Using the formulas for coherent states, we have that

$$\langle 0,\downarrow|e^{\lambda^* a}e^{\lambda a^\dagger}|0,\downarrow\rangle = e^{|\lambda|^2}\langle 0,\downarrow|0,\downarrow\rangle = e^{|\lambda|^2}$$

so $\mathcal{N} = e^{-|\lambda|^2/2}$. Setting the detuning to zero, the time-dependent wavefunction becomes

$$|\psi(t)\rangle = e^{-iH_{JC}t}|\psi(t=0)\rangle = e^{-iH_{JC}t}e^{-\frac{|\lambda|^2}{2}}\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}|n,\downarrow\rangle = e^{-\frac{|\lambda|^2}{2}}\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}(e^{-iH_{JC}t}|n,\downarrow\rangle).$$

The last parenthesis is precisely the time-dependent wavefunction which we used in the previous subproblem. The required time-dependent probability is thus

$$\sum_{n=0}^{\infty}|\langle n,\uparrow|\psi(t)\rangle|^2 = e^{-|\lambda|^2}\frac{1}{2}\sum_{n=0}^{\infty}\frac{|\lambda|^{2n}}{n!}(1-\cos(\Omega\sqrt{n}t)) = \frac{1}{2} - \frac{1}{2}e^{-|\lambda|^2}\sum_{n=0}^{\infty}\frac{|\lambda|^{2n}}{n!}\cos(\Omega\sqrt{n}t).$$

f)* Collapse and revival

The sum contains the factor

$$\frac{|\lambda|^{2n}}{n!} = e^{2n\ln|\lambda| - \ln n!} \simeq e^{2n\ln|\lambda| - n\ln n + n}\frac{1}{\sqrt{2\pi n}}$$

where we have assumed $n >> 1$. The argument in the exponential has a peak at a value n_p such that $2\ln|\lambda| - \ln n_p = 0 \Rightarrow n_p = |\lambda|^2$. Expanding the function $f(n) \equiv 2n\ln|\lambda| - n\ln n + n$ to quadratic order around this point yields

$$f(n) = f(n_p) + \frac{1}{2n_p^2}(n - n_p)^2 + ... = |\lambda|^2 + \frac{1}{2|\lambda|^2}(n - |\lambda|^2)^2 + ...$$

Rewriting the sum in terms of a new symbol m with $n = |\lambda|^2 + m$, m running from $-|\lambda|^2$ to ∞ (we can replace the lower bound by $-\infty$ since the summand is negligible for large $|m|$), we get the required

$$P_{exc}(t) \simeq \frac{1}{2} - \frac{1}{2\sqrt{2\pi|\lambda|^2}} \,\mathrm{Re}\left(\sum_{m=-\infty}^{\infty} e^{-\frac{m^2}{2|\lambda|^2} + i\Omega t\sqrt{|\lambda|^2+m}}\right).$$

The Gaussian form shows us that we only really need to look at contributions from terms with $m \lesssim |\lambda|$. For large $|\lambda|^2$, we can write $\sqrt{|\lambda|^2 + m} \simeq |\lambda| + \frac{m}{2|\lambda|}$. The terms around the peak thus have a time dependence of the form $e^{i\Omega[|\lambda| + \frac{m}{2|\lambda|}]t} \equiv e^{i\Omega_m t}$. These thus oscillate with a frequency $\sim \Omega|\lambda|$, in other words the time scale for oscillations is $T_{osc} \simeq \frac{1}{\Omega|\lambda|}$. Thinking of the sum of terms with $|m| < |\lambda|$, we can see that the phases (which are all the same at $t = 0$) of the terms within this restricted sum, get uniformly distributed over the range 0 to 2π (and thus more or less average out to zero, representing decay of the oscillations) in a time T_{dec} such that $T_{dec}(\Omega_{|\lambda|} - \Omega_0) \simeq 2\pi$, in other words (not caring about constants of order 1) $T_{dec} \simeq 1/\Omega$. These oscillations will however all come in phase together again at a time T_{rev} such that $T_{rev}(\Omega_{m+1} - \Omega_m) \simeq 1$, namely $T_{rev} \simeq |\lambda|/\Omega$.