Statistical Physics & Condensed Matter Theory I: Exercise

The resonant level model

Consider a free Hamiltonian for noninteracting (spinless) fermions moving in one dimension,

$$H_0 = \sum_k \varepsilon_k c_k^\dagger c_k$$

written in terms of annihilation/creation operators $c_k, c_k^\dagger$ for fermions of momentum k. These operators obey canonical anticommutation relations $\{c_k, c_{k'}^\dagger\} = \delta_{k,k'}$.

Let us now imagine that there is an extra 'impurity' site, to which we associate creation/annihilation operators $d, d^\dagger$ (which are also fermionic, so $\{d, d^\dagger\} = 1$). We associate an energy ε_d to the presence of a fermion on the impurity site; we moreover let the fermions hop between the impurity and the line with an amplitude t (which we take to be real), so our full Hamiltonian is

$$H_{RLM} = \sum_k \varepsilon_k c_k^\dagger c_k + \varepsilon_d d^\dagger d + t \left((\sum_k c_k^\dagger) d + d^\dagger \sum_k c_k \right). \tag{1}$$

This is known as the *resonant level* model.

a)

Show explicitly that this Hamiltonian conserves the total number of fermions $N_f = \sum_k c_k^\dagger c_k + d^\dagger d$.

b)

We would like to diagonalize the resonant level Hamiltonian, *i.e.* to obtain a form $H_{RLM} = \sum_n E_n f_n^\dagger f_n + (cst)$, in terms of fermionic operators $f_n, f_n^\dagger$ (again obeying canonical anticommutation relations) which create eigenstates labeled by an index n. Thinking about the case of a *single* fermion, *i.e.* to the sector $N_f = 1$, we can expect to be able (for each individual eigenstate n) to write the creation operator $f_n^\dagger$ as a linear combination of the $c_k^\dagger$ and $d^\dagger$ operator, *i.e.*

$$f_n^\dagger = \sum_k M_{n,k} c_k^\dagger + L_n d^\dagger.$$

This will then indeed be a raising operator for H_{RLM} (in other words: $|n\rangle \equiv f_n^\dagger |0\rangle$ will be an eigenstate of energy E_n, where $|0\rangle$ is the vacuum state for c_k and d (and thus for f_n), *i.e.* $c_k|0\rangle = 0$, $d|0\rangle = 0$) provided we have

$$\left[H_{RLM}, f_n^\dagger\right] = E_n f_n^\dagger.$$

Show that this condition leads to coupled equations for the coefficients $M_{n,k}$ and L_n.

c)

By direct substitution (*e.g.* for $M_{n,k}$ in the expression for L_n), obtain a self-consistency equation for the energies E_n in which only the Hamiltonian parameters $(\varepsilon_k, \varepsilon_d, t)$ appear.

d)

In order to fix the coefficient L_n, we can invoke the normalization requirement of the one-particle state, namely $\langle n|n\rangle = \langle 0|f_n f_n^\dagger|0\rangle = 1$. Using this, obtain an expression for $|L_n|$ in terms of E_n and the Hamiltonian parameters.

e)

To go further, we have to specify the form of the dispersion relation ε_k. For simplicity, we take evenly-spaced levels, and just use integers for momentum, *i.e.*

$$\varepsilon_k = \Delta(k - 1/2), \qquad k \in \mathbb{Z}.$$

Using the identity $\sum_{k\in\mathbb{Z}} \frac{1}{E_n - \varepsilon_k} = -\frac{\pi}{\Delta} \tan \frac{\pi E_n}{\Delta}$ (and its extension $\sum_{k\in\mathbb{Z}} \frac{1}{(E_n-\varepsilon_k)^2} = -\frac{\partial}{\partial E_N} \sum_{k\in\mathbb{Z}} \frac{1}{E_n-\varepsilon_k}$; you might want to remember that $\frac{d}{dx} \tan x = \frac{1}{\cos^2 x}$ and that $\tan^2 x = \frac{1}{\cos^2 x} - 1$), show that the coefficient $|L_n|^2$ takes the form of a *Lorentzian*

$$|L_n|^2 = \frac{I}{(E_n - \alpha)^2 + \gamma^2}$$

and give the value of the coefficients I, α, γ.

For your information: physically, this ultimately means that an initial state with a fermion localized on the impurity will decay into continuum modes with a rate obtainable from the parameters above (the lifetime of such a state is in fact $1/\gamma$).