Statistical Physics & Condensed Matter Theory I: Exercise

Spin waves and the Kubo formula

Consider a one-dimensional lattice of N sites, with spin operators $\mathbf{S}_m$ defined at each site $m = 1, ..., N$, with periodic boundary conditions $\mathbf{S}_{m+N} \equiv \mathbf{S}_m$. We are interested in the ferromagnetic Heisenberg Hamiltonian

$$H = -J \sum_{j=1}^{N} \left[\frac{1}{2} \left(S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+ \right) + \Delta S_j^z S_{j+1}^z \right]$$

with $J > 0$, and in which the anisotropy parameter Δ is an arbitrary real number (we take it here to be positive).

a)

Describe (if possible) all classical and quantum ground states of this system, treating the $\Delta > 1$ and $0 < \Delta < 1$ cases separately.

b)

Using the Holstein-Primakoff transformation, write the effective bosonic theory at large S to leading nontrivial order in $1/S$ (in other words, keep the order S^2 and order S terms but drop the order 1 terms).

c)

Obtain the spectrum of the theory to leading nontrivial order in the $1/S$ expansion. Again for the case $\Delta > 1$, what does the spectrum look like when the momentum is close to zero ? Do you think that this approach also works for $\Delta < 1$? Explain your reasoning.

d)

Let us from now on restrict ourselves to the case $\Delta > 1$. We shall be interested in the space- and time-dependent correlations between the spins (in the large S limit). Since the spin operators are written in terms of bosons, we can build everything in terms of the latter's correlators. Therefore, as a first step, for a free bosonic theory $H_0 = \sum_k \epsilon_k a_k^\dagger a_k$, calculate the retarded correlation function

$$\mathcal{C}^{ret}_{k_1,k_2}(t_1 - t_2) \equiv -i\theta(t_1 - t_2)\langle [a_{k_1}(t_1), a^\dagger_{k_2}(t_2)] \rangle$$

in which the operators are in the interaction representation[1] $a(t) = e^{iH_0 t} a e^{-iH_0 t}$. *Hint: it's easiest to do it directly (i.e. using operators, so without the field integral); you can calculate the zero-temperature correlation (i.e. on the ground state), though the correlator turns out not to depend on which state you're calculating it on.*

[1]... which here is the same as the Heisenberg representation since the perturbation is absent.

e)

Using the Kubo formula (see Useful Formulas), and specializing to zero temperature, calculate the effect (in linear response) of applying the operator $S^x_{j_1}$ at time t_1, on the expectation value of operator $S^x_{j_2}$ at time t_2, to leading order in the large S expansion. *Hint: simply consider applying the time-dependent perturbation $f\delta(t-t_1)S^x_{j_1}$ (with f representing some very small 'probing' amplitude). Remember that $S^x = \frac{1}{2}(S^+ + S^-)$, and that the ground state is fully polarized.* **For your information:** this and similar correlations can be used to describe *inelastic neutron scattering* experiments.

f)*

Go back to the derivation of the Hamiltonian for bosons, and keep the order S^0 term in the $1/S$ expansion. Show that this gives an interaction between the Holstein-Primakoff bosons of the form

$$H_{int} = \frac{1}{N} \sum_{k,k',q} V_{k,k',q} a^\dagger_{k+q} a^\dagger_{k'-q} a_{k'} a_k.$$

Give the explicit form of $V_{k,k',q}$. Considering again $\Delta > 1$ and small momenta, is this interaction repulsive or attractive? What does this mean for the stability of the theory at this order in the $1/S$ expansion?