Statistical Physics & Condensed Matter Theory I: Exercise

Toy path integral

Consider a single fermionic mode described by the operators $a, a^\dagger$, with standard anticommutator $\{a, a^\dagger\} \equiv aa^\dagger + a^\dagger a = 1$. The Hamiltonian is simply given by $H = \varepsilon a^\dagger a$. We have seen that the partition function $\mathcal{Z} = \text{Tr } e^{-\beta H}$ can be written (using fermionic coherent states) as an integral over Grassmann variables,

$$\mathcal{Z} = \text{Tr } e^{-\beta H} = \int d(\bar{\psi}, \psi) e^{-\bar{\psi}\psi} \langle -\psi | e^{-\beta H} | \psi \rangle$$

where $d(\bar{\psi}, \psi) \equiv d\bar{\psi} d\psi$.

a)

We now apply the logic of the path integral. The operator $e^{-\tau H}$ is then viewed as the generator of translations in imaginary time $\tau \in [0, \beta]$. The full imaginary time interval $[0, \beta]$ can thus be split into N equal intervals, and the operator $e^{-\beta H}$ into a product

$$e^{-\beta H} = \left[e^{-\delta H}\right]^N = e^{-\delta H} e^{-\delta H} ... e^{-\delta H} (\text{N times}), \qquad \delta \equiv \beta / N.$$

Consider for simplicity the case $N = 3$. Using resolutions of the identity

$$\int d(\bar{\psi}^n, \psi^n) e^{-\bar{\psi}^n \psi^n} |\psi^n\rangle\langle\psi^n| \equiv 1$$

at time slices separating the different time intervals (in which the superindex ^{n} labels the time slice) and the properties of coherent states, explicitly show that (considering $\delta = \beta/3$ to be small) the partition function can be written as a toy path integral,

$$\mathcal{Z} \simeq \mathcal{Z}_3 = \int \prod_{i=0}^{2} d(\bar{\psi}^i, \psi^i) \text{ exp } \left\{ -(\bar{\psi}^2 \bar{\psi}^1 \bar{\psi}^0) \begin{pmatrix} 1 & -\alpha & 0 \\ 0 & 1 & -\alpha \\ \alpha & 0 & 1 \end{pmatrix} \begin{pmatrix} \psi^2 \\ \psi^1 \\ \psi^0 \end{pmatrix} \right\}$$

where $\alpha \equiv 1 - \delta\varepsilon$ and in which we have denoted $\psi^0 \equiv \psi$ (*Hint:* $e^{-\delta H} = 1 - \delta H + O(\delta^2)$, *but you also have* $1 + \delta x = e^{\delta x} + O(\delta^2)$ *!*).

b)

Explicitly evaluate $\mathcal{Z}_3$, using standard identities for Gaussian integration over Grassmann variables. Express your answer in terms of α.

c)

Generalize the result from part a) to N time intervals and obtain an explicit expression for the associated $\mathcal{Z}_N$. What does the limit $N \to \infty$ yield (*hint:* $\lim_{N\to\infty}(1 + a/N)^N = e^a$) ? Explain how you could have obtained this result for the partition function in one line of calculation.

d)

Consider now a single *bosonic* mode, with Hamiltonian $H = \varepsilon b^\dagger b$, and $[b, b^\dagger] \equiv bb^\dagger - b^\dagger b = 1$. The partition function can then be written in terms of an integral over complex variables,

$$\mathcal{Z} = \mathrm{Tr}\ e^{-\beta H} = \int d(\bar{\psi}, \psi) e^{-\bar{\psi}\psi} \langle \psi | e^{-\beta H} | \psi \rangle$$

where $d(\bar{\psi}, \psi) \equiv \prod_{i=1}^{N} \frac{d\Re\psi_i d\Im\psi_i}{\pi}$. Write an expression for the corresponding $\mathcal{Z}_N$, calculate it, and give the result for the partition function in the limit $N \to \infty$. Is this result what you expected ? (*Hint:* $1 + a + a^2 + a^3 + ... = \frac{1}{1-a}$).