Statistical Physics & Condensed Matter Theory I: Exercise

Tunneling spectroscopy: solution

a)

$$I = i[H, N_2] = i[H_t, N_2] = \sum_{k_1,k_2,k_3} it_{k_1k_2}[a^\dagger_{1,k_1}a_{2,k_2}, a^\dagger_{2,k_3}a_{2,k_3}] + \text{h.c.}$$
$$= \sum_{k_1,k_2,k_3} it_{k_1k_2}a^\dagger_{1,k_1}\{a_{2,k_2}, a^\dagger_{2,k_3}\}a_{2,k_3} + \text{h.c.} = i\sum_{kk'} t_{kk'}a^\dagger_{1,k}a_{2,k'} + \text{h.c.}.$$

b)

$$\mathcal{C}^{I,H_t}_{ret}(t-t') = -i\theta(t-t')\langle\left[I^I(t), H^I_t(t')\right]\rangle = -i\theta(t-t')\left(\langle\left[J^I(t), (T^\dagger)^I(t')\right]\rangle + \langle\left[(J^\dagger)^I(t), T^I(t')\right]\rangle\right)$$

Looking at the first term (the second is its hermitian conjugate), using Wick's theorem and assuming that the correlators are purely diagonal in their indices,

$$\theta(t-t')\sum_{k_1k_2k_3k_4} t_{k_1k_2}t^*_{k_3k_4}\langle\left[a^\dagger_{1,k_1}(t)a_{2,k_2}(t), a^\dagger_{2,k_4}(t')a_{1,k_3}(t')\right]\rangle$$
$$= \theta(t-t')\sum_{k_1k_2k_3k_4} t_{k_1k_2}t^*_{k_3k_4}\Big(\langle a^\dagger_{1,k_1}(t)a_{1,k_3}(t')\rangle_{\beta,\mu_1}\langle a_{2,k_2}(t)a^\dagger_{2,k_4}(t')\rangle_{\beta,\mu_2}$$
$$-\langle a_{1,k_3}(t')a^\dagger_{1,k_1}(t)\rangle_{\beta,\mu_1}\langle a^\dagger_{2,k_4}(t')a_{2,k_2}(t)\rangle_{\beta,\mu_2}\Big)$$
$$= \theta(t-t')\sum_{k_1k_2}|t_{k_1k_2}|^2\Big(\langle a^\dagger_{1,k_1}(t)a_{1,k_1}(t')\rangle_{\beta,\mu_1}\langle a_{2,k_2}(t)a^\dagger_{2,k_2}(t')\rangle_{\beta,\mu_2}$$
$$-\langle a_{1,k_1}(t')a^\dagger_{1,k_1}(t)\rangle_{\beta,\mu_1}\langle a^\dagger_{2,k_2}(t')a_{2,k_2}(t)\rangle_{\beta,\mu_2}\Big)$$

By using the definition of the 'greater' and 'lesser' functions,

$$\mathcal{C}^>_{\beta,\mu;k}(t_1-t_2) = -i\langle a_k(t_1)a^\dagger_k(t_2)\rangle_{\beta,\mu}, \qquad \mathcal{C}^<_{\beta,\mu;k}(t_1-t_2) = i\zeta\langle a^\dagger_k(t_2)a_k(t_1)\rangle_{\beta,\mu}$$

we immediately get that the first term is

$$\theta(t-t')\sum_{k_1k_2}|t_{k_1k_2}|^2\left(-\mathcal{C}^<_{\beta,\mu_1;k_1}(t'-t)\mathcal{C}^>_{\beta,\mu_2;k_2}(t-t') + \mathcal{C}^>_{\beta,\mu_1;k_1}(t'-t)\mathcal{C}^<_{\beta,\mu_2;k_2}(t-t')\right).$$

Putting this in the Kubo formula (shifting the time integration parameter t' by t for convenience) then gives the answer.

c) This is straightforward. The principal part integral can be dropped since we only need the real part.

d) In the low temperature limit, we have that $\lim_{\beta\to\infty}\frac{d}{d\omega}n_F(\omega;\beta) = -\delta(\omega)$. This readily gives the answer.