g)

$$\mathcal{G}_{kn} = \frac{\int \mathcal{D}(\bar{\psi},\psi) e^{-S_0[\bar{\psi},\psi]} \left(1 - S_{int}[\bar{\psi},\psi] + \ldots\right) \bar{\psi}_{kn\sigma}\psi_{kn\sigma}}{\int \mathcal{D}(\bar{\psi},\psi) e^{-S_0[\bar{\psi},\psi]} \left(1 - S_{int}[\bar{\psi},\psi] + \ldots\right)}$$

Rewrite denominator:

$$\text{"Denominator"} = \mathcal{Z}_0 \left(1 - \langle S_{int}\rangle_0 + \ldots\right)$$

Using that $1/(1-x) = 1 + x + \mathcal{O}(x^2)$ for $|x| < 1$,

$$\begin{aligned}
\mathcal{G}_{kn} &= \frac{1}{\mathcal{Z}_0} \int \mathcal{D}(\bar{\psi},\psi) \left[1 - S_{int}[\bar{\psi},\psi] + \ldots\right] \bar{\psi}_{kn\sigma}\psi_{kn\sigma} \left[1 + S_{int}[\bar{\psi},\psi] + \mathcal{O}(U^2)\right] \\
&= \mathcal{G}^{(0)}_{kn} - \left(\langle \bar{\psi}_{kn\sigma}\psi_{kn\sigma} S_{int}[\bar{\psi},\psi]\rangle_0 - \mathcal{G}^{(0)}_{kn}\langle S_{int}\rangle_0\right) + \mathcal{O}(U^2)
\end{aligned}$$

The term within brackets is the first order correction $\mathcal{G}^{(1)}_{kn}$ to the free Green's function; it amounts to taking the connected part of corresponding diagrams, since we have subtracted the disconnected part $\mathcal{G}^{(0)}_{kn}\langle S_{int}\rangle_0$. Doing so, we see that for fixed spin index σ only a single diagram remains. E.g. if σ is 'up', then we have to attach $\bar{\psi}_{kn\sigma}$ to the leg labelled by k, n, and 'up', while $\psi_{kn\sigma}$ has to be attached to the leg labelled by $k+q$, $n+m$ and 'up'. The only way for this diagram to give a non-zero contribution is when $q = m = 0$. The remaining sum over k' and n' remains, and we are left with

$$\begin{aligned}
\mathcal{G}^{(1)}_{kn} &= \frac{UT}{N} \left(\mathcal{G}^{(0)}_{kn}\right)^2 \sum_{k',n'} \mathcal{G}^{(0)}_{k'n'} \\
&= \frac{U}{N} \left(\frac{1}{i\omega_n - \xi_k}\right)^2 \sum_{k'} n_F(\xi_{k'}) \\
&= U \left(\frac{1}{i\omega_n - \xi_k}\right)^2 \frac{k_F}{\pi}
\end{aligned}$$

where we performed the Matsubara sum when going to the second line, and we took the limit $N, \beta \to \infty$ to get to the last line. (Note: when σ is 'down', we have to attach to the primed variables, and we get a sum over k and n, which leads to the same result.)

Hence, the interacting Green's is given by

$$
\begin{aligned}
\mathcal{G}_{kn} &= \mathcal{G}^{(0)}_{kn} + \mathcal{G}^{(1)}_{kn} + \mathcal{O}(U^2) \\
&= \frac{1}{i\omega_n - \xi_k} + \left(\frac{1}{i\omega_n - \xi_k}\right)^2 \frac{Uk_F}{\pi} + \mathcal{O}(U^2) \\
&= \frac{1}{i\omega_n - \xi_k}\left[1 + \frac{Uk_F}{\pi(i\omega_n - \xi_k)}\right] + \mathcal{O}(U^2) \\
&= \frac{1}{i\omega_n - \xi_k}\,\frac{1}{1 - \frac{Uk_F}{\pi(i\omega_n - \xi_k)}} + \mathcal{O}(U^2) \\
&= \frac{1}{i\omega_n - \xi_k - \Sigma_{k,n}} + \mathcal{O}(U^2)
\end{aligned}
$$

where $\Sigma_{k,n} = \frac{Uk_F}{\pi}$ (NB. The sigma doesn't stand for a summation. Perhaps slightly confusing, but standard notation so we're stuck with it.)

h) In the limit $U \to -\infty$, assuming an equal amount of spin-up and spin-down electrons, all particles team up to form a composite bosonic particle to minimize the interaction energy. These composite particles fully occupy a single site, so they behave as free hard core bosons (particles obeying bosonic statistics, but cannot be in the same place because each such particle fully occupies an entire site).