Statistical Physics and Condensed Matter Theory I: Final exam

Monday 21 October 2013, 9:00 - 12:00, SP C1.110

- Please write **legibly** and **be explicit** in your answers. I cannot give you points for things I can't/don't see !
- Please **use separate sheets for each question**, and put your **name, student number and study programme** on each of them.
- There is a collection of useful formulas at the end, class notes and books are **not** allowed.
- This exam consists of 3 problems. You should do **all of them**.
- Problem 3 is identical to the exercise recently given in class.
- Sub-questions marked with * are particularly challenging. Consider solving them only once you're finished with the rest.
- The points add up to 105, so you effectively start with 5 points bonus!

1. Spin waves and the Kubo formula (35 pts)

Consider a one-dimensional lattice of N sites, with spin operators $\mathbf{S}_m$ defined at each site $m = 1, ..., N$, with periodic boundary conditions $\mathbf{S}_{m+N} \equiv \mathbf{S}_m$. We are interested in the ferromagnetic Heisenberg Hamiltonian

$$H = -J \sum_{j=1}^{N} \left[\frac{1}{2} \left(S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+ \right) + \Delta S_j^z S_{j+1}^z \right]$$

with $J > 0$, and in which the anisotropy parameter Δ is an arbitrary real number (we take it here to be positive).

a) (5 pts)

Describe (if possible) all classical and quantum ground states of this system, treating the $\Delta > 1$ and $0 < \Delta < 1$ cases separately.

b) (5 pts)

Using the Holstein-Primakoff transformation, write the effective bosonic theory at large S to leading nontrivial order in $1/S$ (in other words, keep the order S^2 and order S terms but drop the order 1 terms).

c) (10 pts)

Obtain the spectrum of the theory to leading nontrivial order in the $1/S$ expansion. Again for the case $\Delta > 1$, what does the spectrum look like when the momentum is close to zero ? Do you think that this approach also works for $\Delta < 1$? Explain your reasoning.

d) (5 pts)

Let us from now on restrict ourselves to the case $\Delta > 1$. We shall be interested in the space- and time-dependent correlations between the spins (in the large S limit). Since the spin operators are written in terms of bosons, we can build everything in terms of the latter's correlators. Therefore, as a first step, for a free bosonic theory $H_0 = \sum_k \epsilon_k a_k^\dagger a_k$, calculate the retarded correlation function

$$\mathcal{C}^{ret}_{k_1,k_2}(t_1 - t_2) \equiv -i\theta(t_1 - t_2)\langle [a_{k_1}(t_1), a^\dagger_{k_2}(t_2)]\rangle$$

in which the operators are in the interaction representation[1] $a(t) = e^{iH_0 t} a e^{-iH_0 t}$. *Hint: it's easiest to do it directly (i.e. using operators, so without the field integral); you can calculate the zero-temperature correlation (i.e. on the ground state), though the correlator turns out not to depend on which state you're calculating it on.*

e) (5 pts)

Using the Kubo formula (see Useful Formulas), and specializing to zero temperature, calculate the effect (in linear response) of applying the operator $S^x_{j_1}$ at time t_1, on the expectation value of operator $S^x_{j_2}$ at time t_2. *Hint: simply consider applying the time-dependent perturbation $f\delta(t - t_1)S^x_{j_1}$ (with f representing some very small 'probing' amplitude). Remember that $S^x = \frac{1}{2}(S^+ + S^-)$, and that the ground state is fully polarized.* **For your information:** this and similar correlations can be used to describe *inelastic neutron scattering* experiments.

f)* (5 pts)

Go back to the derivation of the Hamiltonian for bosons, and keep the order S^0 term in the $1/S$ expansion. Show that this gives an interaction between the Holstein-Primakoff bosons of the form

$$H_{int} = \frac{1}{N} \sum_{k,k',q} V_{k,k',q} a^\dagger_{k+q} a^\dagger_{k'-q} a_{k'} a_k.$$

Give the explicit form of $V_{k,k',q}$. Considering again $\Delta > 1$ and small momenta, is this interaction repulsive or attractive? What does this mean for the stability of the theory at this order in the $1/S$ expansion?

[1]... which here is the same as the Heisenberg representation since the perturbation is absent.

2. The Jaynes-Cummings model (35 pts)

One of the most important recent breakthroughs in physics has been the ability to isolate and manipulate *single* atoms[2], this being done either in harmonic traps or optical cavities. The central model in this field is called the **Jaynes-Cummings** model, and describes a two-level atom interacting with the quantized modes of a harmonic oscillator (see the figure). This problem uses second quantization to extract some interesting physics from this model.

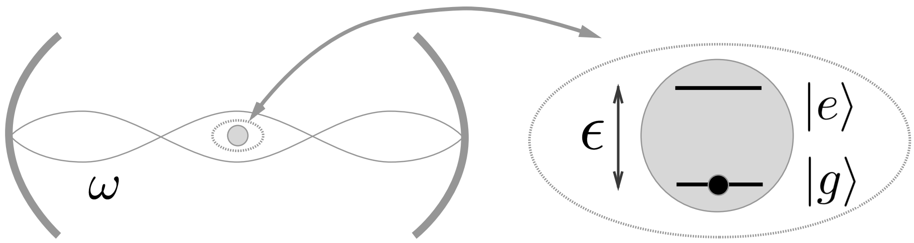

Figure 1: Example setup for the Jaynes-Cummings model. An optical cavity contains quantized modes of frequency ω. A two-level system (*e.g.* two internal states of an atom) with energy splitting ϵ is coupled to the cavity mode. The two-level system is here pictured in its ground state $|g\rangle$. Absorption of a quantum of the cavity mode promotes it to the excited state $|e\rangle$.

The Hamiltonian of the model is

$$H_{JC} = \frac{\epsilon}{2}\sigma^z + \omega a^\dagger a + \frac{\Omega}{2}(a\sigma^+ + a^\dagger \sigma^-).$$

The first term represents the two-level atom, the second represents the cavity mode (a quantum harmonic oscillator), and the third term is the coupling between these. We have represented the two-level atom as a pseudo-spin (denoting the ground state and excited state of the two-level system respectively as $|g\rangle$ and $|e\rangle$)

$$|g\rangle \equiv |\downarrow\rangle, \qquad |e\rangle \equiv |\uparrow\rangle,$$

with operators

$$\sigma^z = |e\rangle\langle e| - |g\rangle\langle g|, \quad \sigma^+ = |e\rangle\langle g|, \quad \sigma^- = |g\rangle\langle e|.$$

The Ω term means that the two-level atom can go from the ground to the excited state by absorbing one quantum of the cavity mode, and conversely relax from the excited to the ground state by emitting a quantum of the cavity mode. The set of states $|n, \uparrow(\downarrow)\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0, \uparrow(\downarrow)\rangle$, $n \in \mathbb{N}$ forms a basis for the Hilbert space.

a) (5 pts)

Show that the quantity

$$\hat{N} = a^\dagger a + \frac{1}{2}\sigma^z$$

commutes with the Jaynes-Cummings Hamiltonian, and is thus a conserved quantity.

b) (10 pts)

Let us now work in a fixed subspace of the Hilbert space in which the operator $\hat{N}$ takes on a definite value n. This subspace is two-dimensional, and we write its two basis states as

$$|n-1, \uparrow\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |n, \downarrow\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

[2]The 2012 Nobel Prize in Physics was awarded to Serge Haroche and David Wineland *'for ground-breaking experimental methods that enable measuring and manipulation of individual quantum systems'*.

Show that when projected onto this subspace, the Hamiltonian is represented as

$$H_{JC}^{(n)} = \begin{pmatrix} \langle n-1,\uparrow|H_{JC}|n-1,\uparrow\rangle & \langle n-1,\uparrow|H_{JC}|n,\downarrow\rangle \\ \langle n,\downarrow|H_{JC}|n-1,\uparrow\rangle & \langle n,\downarrow|H_{JC}|n,\downarrow\rangle \end{pmatrix} = \omega(n-\frac{1}{2})\mathbf{1} + \frac{1}{2}\begin{pmatrix} \delta & \Omega\sqrt{n} \\ \Omega\sqrt{n} & -\delta \end{pmatrix},$$

the first term being proportional to the unit matrix (and thus representing a trivial constant energy shift for given n), and the parameter $\delta \equiv \epsilon - \omega$ being called the *detuning*.

c) (5 pts)

Diagonalize this using a Bogoliubov transformation (you directly use the Useful Formulas below, without rederivation). Give the explicit form of the two eigenstates together with their energy.

d) Rabi oscillations (5 pts)

Consider now an initial state

$$|\psi(t=0)\rangle = |n,\downarrow\rangle.$$

Show that the propability $P_{exc}(t)$ of finding the system in the excited state $|n-1,\uparrow\rangle$ displays **Rabi oscillations** at frequency ω_R (to be determined) according to the formula

$$P_{exc}(t) = \frac{1}{2}(1-\cos\omega_R t)\frac{\Omega^2 n}{\Omega^2 n + \delta^2}.$$

e) Coherent initial state (5 pts)

Let us now consider a more interesting initial state where the bosons are in a *coherent state*:

$$|\psi(t=0)\rangle = \mathcal{N}e^{\lambda a^\dagger}|0,\downarrow\rangle, \qquad \lambda \in \mathbb{C}.$$

Setting the value of $\mathcal{N}$ so that the state is normalized, and considering the simplest case of zero detuning $\delta = 0$, show that the probability of finding the atom in the excited state (*i.e.* the two-level system in state $\uparrow$, irrespective of the number of quanta in the cavity mode) is given by

$$P_{exc}(t) = \frac{1}{2} - \frac{1}{2}e^{-|\lambda|^2}\sum_{n=0}^{\infty}\frac{|\lambda|^{2n}}{n!}\cos(\Omega\sqrt{n}t).$$

f)* Collapse and revival (5 pts)

Consider now the case $|\lambda| \gg 1$ (in other words, the cavity mode is very highly occupied). By using Stirling's formula $\ln(n!) = n\ln n - n + \ln\sqrt{2\pi n} + O(1/n)$ (for $n \gg 1$), argue that the summand is sharply peaked around a value $n_p \simeq |\lambda|^2$, and that by expanding around this peak value we can write the approximate result

$$P_{exc}(t) \simeq \frac{1}{2} - \frac{1}{2\sqrt{2\pi|\lambda|^2}}\,\mathrm{Re}\left(\sum_{m=-\infty}^{\infty} e^{-\frac{m^2}{2|\lambda|^2} + i\Omega t\sqrt{|\lambda|^2+m}}\right).$$

By looking at contributions around the peak at $m = 0$, looking at the set of 'fast' oscillations at frequency $\Omega|\lambda|$, and considering that the most important contributions come from values of m such that $|m| < |\lambda|$ (because others are suppressed by the Gaussian form), argue that there are three relevant time scales to the problem: a time scale for *oscillations*, one for *decay*, and one for *revival*, and that these are respectively given by

$$T_{osc} \simeq \frac{1}{\Omega|\lambda|}, \qquad T_{dec} \simeq \frac{1}{\Omega}, \qquad T_{rev} \simeq \frac{|\lambda|}{\Omega}.$$

For your information: the revival is a purely quantum effect (it would not exist if the cavity modes were not quantized), and is observable experimentally in one-atom masers.

3. Itinerant electrons with interactions: mean-field theory (35 pts)

We have seen that starting from Hubbard-like models (with on-site Coulomb interaction $U > 0$), a Heisenberg model can be obtained by going to the strongly-interacting limit $U \to \infty$ when the system is half-filled (one electron per site on average).

What happens if U is not that large, and we're working at generic fillings? Consider a d-dimensional crystal with electrons interacting with a purely on-site Coulomb interaction $U > 0$. In Fourier space, we can write the Hamiltonian as

$$H = H_0 + H_{int} = \sum_{\mathbf{k}} \sum_{\sigma} \varepsilon_{\mathbf{k}} a^\dagger_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma} + \frac{U}{2L^d} \sum_{\mathbf{k}\mathbf{k}'\mathbf{q}} \sum_{\sigma\sigma'} a^\dagger_{\mathbf{k}+\mathbf{q}\sigma} a^\dagger_{\mathbf{k}'-\mathbf{q}\sigma'} a_{\mathbf{k}'\sigma'} a_{\mathbf{k}\sigma}$$

in which $\varepsilon_{\mathbf{k}}$ is the kinetic energy part.

The interaction term is impossible to handle exactly. We could simply do perturbation theory in U, but we don't always want to assume that U is small. The purpose of this exercise is to show you another way of handling this interaction.

The key is to find a reasonable way of rewriting the interaction term (containing 4 operators) into terms with 2 operators only (which can then be handled exactly). We thus make the assumption ('mean-field') that we can replace the operator product according to

$$\begin{aligned} a^\dagger_{\mathbf{k}+\mathbf{q}\sigma} a^\dagger_{\mathbf{k}'-\mathbf{q}\sigma'} a_{\mathbf{k}'\sigma'} a_{\mathbf{k}\sigma} \simeq{} & a^\dagger_{\mathbf{k}+\mathbf{q}\sigma} a_{\mathbf{k}\sigma} \langle a^\dagger_{\mathbf{k}'-\mathbf{q}\sigma'} a_{\mathbf{k}'\sigma'} \rangle_{MF} + a^\dagger_{\mathbf{k}'-\mathbf{q}\sigma'} a_{\mathbf{k}'\sigma'} \langle a^\dagger_{\mathbf{k}+\mathbf{q}\sigma} a_{\mathbf{k}\sigma} \rangle_{MF} \\ & - a^\dagger_{\mathbf{k}+\mathbf{q}\sigma} a_{\mathbf{k}'\sigma'} \langle a^\dagger_{\mathbf{k}'-\mathbf{q}\sigma'} a_{\mathbf{k}\sigma} \rangle_{MF} - a^\dagger_{\mathbf{k}'-\mathbf{q}\sigma'} a_{\mathbf{k}\sigma} \langle a^\dagger_{\mathbf{k}+\mathbf{q}\sigma} a_{\mathbf{k}'\sigma'} \rangle_{MF} \\ & - \langle a^\dagger_{\mathbf{k}+\mathbf{q}\sigma} a_{\mathbf{k}\sigma} \rangle_{MF} \langle a^\dagger_{\mathbf{k}'-\mathbf{q}\sigma'} a_{\mathbf{k}'\sigma'} \rangle_{MF} + \langle a^\dagger_{\mathbf{k}+\mathbf{q}\sigma} a_{\mathbf{k}'\sigma'} \rangle_{MF} \langle a^\dagger_{\mathbf{k}'-\mathbf{q}\sigma'} a_{\mathbf{k}\sigma} \rangle_{MF} \end{aligned}$$

in which we take the 'mean-field' expectation values to be given by the (not yet determined) parameters $n_{\mathbf{k}\sigma}$ according to

$$\langle a^\dagger_{\mathbf{k}\sigma} a_{\mathbf{k}'\sigma} \rangle_{MF} \equiv \delta_{\mathbf{k}\mathbf{k}'} \bar{n}_{\mathbf{k}\sigma}$$

(all other expectation values vanishing).

a) (10 pts)

Show that under this mean-field assumption, the interaction part of the Hamiltonian is replaced by

$$H_{int} \simeq H^{MF}_{int} = U \sum_{\mathbf{k}} \sum_{\sigma\sigma'} a^\dagger_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma} \left[\bar{n}_{\sigma'} - \delta_{\sigma\sigma'} \bar{n}_\sigma \right] - \frac{UL^d}{2} \sum_{\sigma\sigma'} (1 - \delta_{\sigma\sigma'}) \bar{n}_\sigma \bar{n}_{\sigma'}$$

in which

$$\bar{n}_\sigma \equiv \frac{1}{L^d} \sum_{\mathbf{k}} \langle a^\dagger_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma} \rangle_{MF}$$

are again fixed (though unspecified as of yet) numbers.

b) (10 pts)

The complete mean-field Hamiltonian is thus

$$H^{MF} = H^0 + H^{MF}_{int} \equiv \sum_{\mathbf{k}} \sum_{\sigma} \varepsilon^{MF}_{\mathbf{k}\sigma} a^\dagger_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma} + C(\{\bar{n}_\sigma\})$$

with

$$\varepsilon^{MF}_{\mathbf{k}\sigma} \equiv \varepsilon_{\mathbf{k}} + U(\bar{n}_\uparrow + \bar{n}_\downarrow - \bar{n}_\sigma) = \varepsilon_{\mathbf{k}} + U\bar{n}_{-\sigma}, \qquad C(\{\bar{n}_\sigma\}) \equiv -\frac{UL^d}{2} \sum_{\sigma\sigma'} (1 - \delta_{\sigma\sigma'}) \bar{n}_\sigma \bar{n}_{\sigma'}$$

Since this Hamiltonian is now bilinear in the $a^\dagger$, a operators, everything can be computed exactly. Write down the coherent state path integral representation for the partition function Z^{MF} of the mean-field theory, introducing separate chemical potentials μ_σ for up and down spins, and show that the mean-field free energy can be written

$$\mathcal{F}^{MF} = -T \ln \mathcal{Z}^{MF} = -T \sum_{\mathbf{k}} \sum_{n} \sum_{\sigma} \ln \left[\beta(-i\omega_n + \xi^{MF}_{\mathbf{k}\sigma})\right] + C(\{\bar{n}_\sigma\})$$

in which $\xi^{MF}_{\mathbf{k}\sigma} = \varepsilon^{MF}_{\mathbf{k}\sigma} - \mu_\sigma$.

c) (5 pts)

So far, we have assumed that the mean-field parameters $\bar{n}_\sigma$ were fixed, but we didn't specify to which value. This is done by requiring *self-consistency* of the mean-field treatment. Using the relations $\bar{n}_\sigma = -\frac{1}{L^d}\frac{\partial}{\partial \mu_\sigma}\mathcal{F}^{MF}$, show that (by performing the Matsubara summation using one of the Useful Formulas) we need to require

$$\bar{n}_\sigma = \frac{1}{L^d} \sum_{\mathbf{k}} n_F(\varepsilon^{MF}_{\mathbf{k}\sigma})$$

d) (5 pts)

Specialize now to the three-dimensional case at zero temperature, assume that $\varepsilon_{\mathbf{k}} = \frac{\hbar^2 \mathbf{k}^2}{2m}$, go to the infinite-size limit (so that $\frac{1}{L^3}\sum_{\mathbf{k}}(...) \to \int \frac{d^3k}{(2\pi)^3}(...)$), and take the two chemical potentials to be equal $\mu_\sigma \equiv \mu$. Show that

$$\bar{n}_\sigma = \frac{1}{6\pi^2} k^3_{F\sigma}$$

where the (spin-dependent) Fermi momenta are given by $\frac{\hbar^2}{2m}k^2_{F\sigma} + U\bar{n}_{-\sigma} = \mu$.

e) (5 pts)

Defining the parameters

$$\bar{n} \equiv \bar{n}_\uparrow + \bar{n}_\downarrow, \qquad \zeta = \frac{\bar{n}_\uparrow - \bar{n}_\downarrow}{\bar{n}_\uparrow + \bar{n}_\downarrow}, \qquad \gamma = \frac{2mU\bar{n}^{1/3}}{(3\pi^2)^{2/3}\hbar^2}$$

(note that we must have $0 \le \zeta \le 1$), show that the self-consistency conditions can be rewritten (most easily by subtracting the two chemical potentials from each other)

$$\gamma = \frac{1}{\zeta}\left[(1+\zeta)^{2/3} - (1-\zeta)^{2/3}\right].$$

Discuss what happens as a function of the effective interaction parameter γ (hint: look at the left-hand side function of ζ, and find the limits when $\zeta \to 0$ and $\zeta \to 1$). What kind of magnetic state exists for $\gamma < 4/3$, $4/3 < \gamma < 2^{2/3}$ and $\gamma > 2^{2/3}$? Can you explain this physically?

Useful Formulas

Trigonometric and hyperbolic functions

$$\sin(\theta_1+\theta_2)=\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2,\qquad \cos(\theta_1+\theta_2)=\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2,$$
$$\cos^2\theta+\sin^2\theta=1,\qquad \sin^2\theta=\frac{1}{2}(1-\cos 2\theta),\qquad \cos^2\theta=\frac{1}{2}(1+\cos 2\theta),$$
$$\sinh(\theta_1+\theta_2)=\sinh\theta_1\cosh\theta_2+\cosh\theta_1\sinh\theta_2,\qquad \cosh(\theta_1+\theta_2)=\cosh\theta_1\cosh\theta_2+\sinh\theta_1\sinh\theta_2,$$
$$\cosh^2\theta-\sinh^2\theta=1,\qquad \sinh^2\theta=\frac{1}{2}(\cosh 2\theta-1),\qquad \cosh^2\theta=\frac{1}{2}(\cosh 2\theta+1).$$

Series expansions

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!},\qquad \cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!},\qquad \sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!},$$
$$(1+x)^\alpha=\sum_{n=0}^{\infty}\left(\begin{array}{c}\alpha\\ n\end{array}\right)x^n=1+\alpha x+\frac{\alpha(\alpha-1)}{2}x^2+...,\qquad \ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$$

Bosonic occupation number states

$$[b,b^\dagger]=1,\qquad |n\rangle=\frac{1}{\sqrt{n!}}(b^\dagger)^n|0\rangle,\qquad b^\dagger|n\rangle=\sqrt{n+1}|n+1\rangle,\qquad b|n\rangle=\sqrt{n}|n-1\rangle.$$

Pauli spin matrices

$$\sigma^x=\left(\begin{array}{cc}0&1\\1&0\end{array}\right),\qquad \sigma^y=\left(\begin{array}{cc}0&-i\\i&0\end{array}\right),\qquad \sigma^z=\left(\begin{array}{cc}1&0\\0&-1\end{array}\right),\qquad \sigma^\pm=\frac{1}{2}(\sigma^x\pm i\sigma^y).$$

Spins on a lattice

$SU(2)$ spin algebra (here, $i,j,k=x,y,z$ and m,n denote lattice sites).

$$\left[\hat{S}^i_m,\hat{S}^j_n\right]=i\delta_{mn}\varepsilon^{ijk}\hat{S}^k_n.$$

Spin raising and lowering operators: $\hat{S}^\pm_m=\hat{S}^x_m\pm i\hat{S}^y_m$ with

$$\left[\hat{S}^z_m,\hat{S}^\pm_n\right]=\pm\delta_{nm}\hat{S}^\pm_m,\qquad \left[\hat{S}^+_m,\hat{S}^-_n\right]=2\delta_{nm}\hat{S}^z_m.$$

For the $S=1/2$ case, one can use the representation $S^i=\sigma^i/2$, $i=x,y,z$.

Holstein-Primakoff transformation

$$\hat{S}^-_m=a^\dagger_m(2S-a^\dagger_m a_m)^{1/2},\qquad \hat{S}^+_m=(2S-a^\dagger_m a_m)^{1/2}a_m,\qquad \hat{S}^z_m=S-a^\dagger_m a_m$$

where a_m, $a^\dagger_m$ are bosonic operators obeying the canonical algebra $[a_m,a^\dagger_n]=\delta_{mn}$ (other commutators vanish).

Fourier transformation

$$a_k=\frac{1}{\sqrt{N}}\sum_{m=1}^{N}e^{ikm}a_m,\quad a_m=\frac{1}{\sqrt{N}}\sum_{k\in BZ}e^{-ikm}a_k,\quad [a_k,a^\dagger_{k'}]_\zeta=\left\{\begin{array}{ll}a_k a^\dagger_{k'}-a^\dagger_{k'}a_k, & \text{bosons}\\ a_k a^\dagger_{k'}+a^\dagger_{k'}a_k, & \text{fermions}\end{array}\right.=\delta_{kk'}$$

Bogoliubov transformation

The matrix

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

(here for $a, b \in \mathbb{R}$) can be diagonalized by the unitary transformation

$$UHU^\dagger = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix}, \qquad U = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

where $\tan 2\theta = \frac{b}{a}$ and $\varepsilon = (a^2 + b^2)^{1/2}$.

Coherent states (bosons: $\zeta = 1$, fermions: $\zeta = -1$)

$$|\phi\rangle \equiv \exp\left[\zeta \sum_i \phi_i a_i^\dagger\right] |0\rangle$$

$$a_i|\phi\rangle = \phi_i|\phi\rangle, \qquad a_i^\dagger|\phi\rangle = \zeta\partial_{\phi_i}|\phi\rangle, \qquad \langle\phi|a_i^\dagger = \langle\phi|\bar{\phi}_i, \qquad \langle\phi|a_i = \partial_{\bar{\phi}_i}\langle\phi| \qquad \forall i.$$

The norm of a coherent state is

$$\langle\phi|\phi\rangle = \exp\left[\sum_i \bar{\phi}_i\phi_i\right].$$

Coherent states form an (over)complete set of states:

$$\int \prod_i d(\bar{\phi}_i, \phi_i) e^{-\sum_i \bar{\phi}_i\phi_i} |\phi\rangle\langle\phi| = \mathbf{1}_{\mathcal{F}}$$

with $\mathbf{1}_{\mathcal{F}}$ the identity in Fock space. The measures are $d(\bar{\phi}_i, \phi_i) = \frac{d\bar{\phi}_i d\phi_i}{\pi}$ for bosons, $d(\bar{\phi}_i, \phi_i) = d\bar{\phi}_i d\phi_i$ for fermions.

Campbell-Baker-Hausdorff formula

The general identity called the Campbell-Baker-Hausdorff formula reads:

$$e^{-B}Ae^{B} = \sum_{n=0}^{\infty} \frac{1}{n!}[A, B]_n, \qquad \text{where } [A, B]_n = [[A, B]_{n-1}, B], \quad [A, B]_0 \equiv A.$$

This can be specialized to some simpler particular cases. Let A and B be two quantum operators such that $[A, B]$ commutes with A and B. Then, the following identities hold:

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}, \qquad [A, e^{\lambda B}] = \lambda[A, B]e^{\lambda B}.$$

Another useful one is:

$$\text{if } [A, B] = DB \text{ and } [A, D] = 0 = [B, D], \text{ then } f(A)B = Bf(A + D).$$

This then implies (under the same conditions) that

$$e^A B e^{-A} = Be^D.$$

Grassmann variables

$$\forall i, j, \quad \eta_i\eta_j = -\eta_j\eta_i, \qquad \int d\eta_i = 0, \qquad \int d\eta_i\eta_i = 1.$$

Coherent state path integral representation of the partition function

For a second-quantized Hamiltonian of the form

$$\hat{H}(a^\dagger, a) = \sum_{ij} h_{ij} a_i^\dagger a_j + \sum_{ijkl} V_{ijkl} a_i^\dagger a_j^\dagger a_k a_l,$$

the partition function is

$$\mathcal{Z} = \int \mathcal{D}(\bar{\psi}, \psi) e^{-S[\bar{\psi}, \psi]}.$$

Here, we work directly in the Matsubara frequency (usually labeled by the index n, whose value runs over all integers) representation. The measure is defined as $\mathcal{D}(\bar{\psi}, \psi) = \prod_i \prod_n d(\bar{\psi}_{in}, \psi_{in})$ and $d(\bar{\psi}, \psi) \equiv \beta d\bar{\psi} d\psi$ for fermions and $d(\bar{\psi}, \psi) \equiv \frac{1}{\pi\beta} d\bar{\psi} d\psi$ for bosons (see next subsection for the Gaussian integral). The effective action is

$$S[\bar{\psi}, \psi] = \sum_{ij,n} \bar{\psi}_{in} \left[(-i\omega_n - \mu)\delta_{ij} + h_{ij}\right] \psi_{jn} + T \sum_{ijkl,\{n_i\}} V_{ijkl} \bar{\psi}_{in_1} \bar{\psi}_{jn_2} \psi_{kn_3} \psi_{ln_4} \delta_{n_1+n_2,n_3+n_4}.$$

Gaussian integration over bosonic/Grassmann variables

By definition, in the frequency representation of the action, we use

$$\int d(\bar{\psi}, \psi) e^{-\bar{\psi}\varepsilon\psi} = (\beta\varepsilon)^{-\zeta}$$

with $\zeta = +1$ for bosons and -1 for fermions.

Wick's theorem (fermions)

The expectation value of a product of fermionic fields over a noninteracting theory is given by the sum over all pairings signed by the permutation order. For four fields,

$$\langle \bar{\psi}_a \bar{\psi}_b \psi_c \psi_d \rangle_0 = \langle \bar{\psi}_a \psi_d \rangle_0 \langle \bar{\psi}_b \psi_c \rangle_0 - \langle \bar{\psi}_a \psi_c \rangle_0 \langle \bar{\psi}_b \psi_d \rangle_0.$$

The first term is the Hartree term, the second is the Fock term.

Matsubara sums (fermions)

$$\sum_n \ln(\beta\left[-i\omega_n + \xi\right]) = \ln\left[1 + e^{-\beta\xi}\right],$$

$$T \sum_n \frac{1}{i\omega_n - \varepsilon_a + \mu} = \frac{1}{e^{\beta(\varepsilon_a - \mu)} + 1} \equiv n_F(\varepsilon_a, \mu).$$

Interaction representation

For the Hamiltonian $H = H_0 + H_I$ in which H_I represents the 'interaction' and H_0 the free (exactly-solvable) model, the interaction picture states and operators are related to the Schrödinger ones by

$$|\psi^I(t)\rangle = e^{iH_0 t}|\psi^S(t)\rangle, \qquad \mathcal{O}^I(t) = e^{iH_0 t}\mathcal{O}^S e^{-iH_0 t}.$$

Linear response theory: the Kubo formula

For the time-dependent Hamiltonian (in the Schrödinger picture)

$$H(t) = H_0 + F(t)\hat{P},$$

with initial condition that the system at $t \to -\infty$ is in state $|\psi_o\rangle$, the time-dependent expectation value of operator $\mathcal{O}$ is given in linear response by the Kubo formula

$$\bar{O}(t) = \langle\psi_0|\hat{O}|\psi_0\rangle + \int_{-\infty}^{\infty} dt' \mathcal{C}^{\hat{O},\hat{P}}_{ret,\psi_0}(t-t')F(t') + O(F^2)$$

in terms of the retarded correlation function (computed in state $|\psi_0\rangle$) between the perturbation and observable, this retarded function being defined (for a generic state $|\psi\rangle$) as

$$\mathcal{C}^{\hat{O},\hat{P}}_{ret,\psi}(t-t') \equiv -i\theta(t-t')\langle\psi|[\hat{O}^I(t), \hat{P}^I(t')]|\psi\rangle.$$