Statistical Physics and Condensed Matter Theory I: Final exam

Thursday 22 October 2015, 9:00 - 12:00, REC C1.04

- Please write **legibly** and **be explicit** in your answers. I cannot give you points for things I can't/don't see !
- Please **use separate sheets for each question**, and put your **name, student number and study programme** on each of them.
- There is a collection of useful formulas at the end, which you can use **without rederivation**. Class notes and books are **not** allowed.
- This exam consists of 3 problems. You should do **all of them**.
- Sub-questions marked with * are particularly challenging. Consider solving them only once you're finished with the rest.
- **Be smart**: if you're stuck on a (sub-)question, don't lose too much time, you can always move on to the next one (the questions are formulated in order to make this possible).
- The points add up to 110, that's 10% bonus for you from the start.

1. Stellar photon escape time (10 pts)

Most of you know that the light we get from the Sun has only taken a little more than 8 minutes to travel from the Sun's surface all the way to the Earth. These photons however did not begin their journey at the Sun's surface. They of course originate from nuclear reactions near the core (center); these photons must thus first propagate from there to the surface. Due to the large scattering cross section for electron-photon scattering, the photons, after being 'born', really embark on a long random walk to the surface (and thus freedom).

How long does it take for a photon to reach the surface of the Sun, if it's created at the center? Give an order-of-magnitude estimate for this escape time using what you know of random walks (see **Useful Formulas** for a reminder; I do *not* expect to see long calculations). You will need the following numbers: Sun's radius ~ 700000 km, electron-photon collision time (*i.e.:* mean time between collisions) $\sim 10^{-10}$ s, and of course speed of light ~ 300000 km/s.

2. Jordan-Wigner transformation; the XY model (60 pts)

Consider an isolated spin-1/2 quantum degree of freedom. The spin operators S^z and $S^\pm$ ($S^\pm = S^x \pm iS^y$) associated to this obey the $su(2)$ algebra

$$\left[S^z, S^\pm\right] = \pm S^\pm, \qquad \left[S^+, S^-\right] = 2S^z. \tag{1}$$

For the case of spin-1/2, the representation is two-dimensional with the two base states $|\pm\rangle$ such that

$$S^z|\pm\rangle = \pm\frac{1}{2}|\pm\rangle, \quad S^+|+\rangle = 0, \quad S^-|+\rangle = |-\rangle, \quad S^+|-\rangle = |+\rangle, \quad S^-|-\rangle = 0.$$

Another obvious realization of a two-state quantum system is to consider a fermionic degree of freedom, whose annihilation and creation operators we write as $c, c^\dagger$ (these obeying the canonical anticommutation relation $\{c, c^\dagger\} = 1$, with other anticommutators vanishing). The space of states is spanned by two base states, $|0\rangle$ and $|1\rangle = c^\dagger|0\rangle$, where $|0\rangle$ is the vacuum of c: $c|0\rangle = 0$.

Since the space dimensionalities coincide, can we go further and explicitly map spins to/from fermions?

a) (5 pts) Let us choose to associate 'spin up' with 'no fermion', and 'spin down' with 'one fermion'. Show that rewriting the fermion operators in terms of spins according to

$$S^z = \frac{1}{2} - c^\dagger c, \qquad S^+ = c, \qquad S^- = c^\dagger \tag{2}$$

allows to reproduce the spin algebra (1) from the fermionic anticommutator algebra.

b) (10 pts) Let us now consider the slightly more complicated case of *two* spin-1/2 degrees of freedom, with operators S_j, $j = 1, 2$ obeying the multiple-site $su(2)$ algebra:

$$\left[S^z_j, S^z_{j'}\right] = 0, \qquad \left[S^z_j, S^\pm_{j'}\right] = \pm\delta_{jj'}S^\pm_j, \qquad \left[S^+_j, S^-_{j'}\right] = 2\delta_{jj'}S^z_j. \tag{3}$$

Namely, the spin operators *commute* on different sites. The space of states is then spanned by $|\sigma_1, \sigma_2\rangle$ with $\sigma_i = \pm$.

We now also consider fermions on two sites, with operators $c_j, c_j^\dagger, j = 1, 2$ obeying the canonical *anti*commutation relations on multiple sites:

$$\left\{c_j, c_{j'}^\dagger\right\} = \delta_{jj'}, \qquad \{c_j, c_{j'}\} = 0, \qquad \left\{c_j^\dagger, c_{j'}^\dagger\right\} = 0. \tag{4}$$

We however hit a slight problem if we naively extend our previous single-site result (2) for the fermion-spin operator mapping to two sites: for example,

$$S_j^+ \stackrel{?}{=} c_j, \quad S_j^- \stackrel{?}{=} c_j^\dagger: \qquad \left[S_1^+, S_2^-\right] = c_1 c_2^\dagger - c_2^\dagger c_1 \neq 0.$$

The mismatch between the canonical commutation of spin operators, and anticommutation of fermionic operators thus makes a simple local mapping between spins and fermions impossible.

Show that the following slightly more complicated mapping, in which we 'dress' fermionic operators on site 2 with a 'tail' involving operators to the left (*i.e.* on site 1), allows to reproduce the spin algebra (3) from the fermionic one (4):

$$\begin{aligned}
S_1^z &= \frac{1}{2} - \hat{n}_1, \quad & S_1^+ &= c_1, \quad & S_1^- &= c_1^\dagger, \\
S_2^z &= \frac{1}{2} - \hat{n}_2, \quad & S_2^+ &= (1 - 2\hat{n}_1)c_2, \quad & S_2^- &= (1 - 2\hat{n}_1)c_2^\dagger,
\end{aligned}$$

in which we used the shorthand notation $\hat{n}_j \equiv c_j^\dagger c_j$.

Note that the factor $1 - 2\hat{n}_1$ can only take values ± 1. It can also equivalently be written as $2S_1^z$ or $e^{\pm i\pi\hat{n}_1}$.

c) (10 pts) This idea immediately generalizes to an arbitrary number of spin-1/2 operators S_j^a with a definite ordering $j = 1, 2, ...$ In this general setting, it is known as the **Jordan-Wigner transformation**

$$\textbf{Jordan-Wigner:} \qquad S_j^z = \frac{1}{2} - \hat{n}_j, \quad S_j^+ = \left[\prod_{l=1}^{j-1}(1 - 2\hat{n}_l)\right] c_j, \quad S_j^- = \left[\prod_{l=1}^{j-1}(1 - 2\hat{n}_l)\right] c_j^\dagger.$$

Show that indeed, under this mapping, the fermionic algebra on multiple sites implies the $su(2)$ algebra on multiple sites (referring to the algebra definitions above).

d) (10 pts) A particularly important model in magnetism is the so-called **XY model**. Consider thus a one-dimensional chain of N sites, with spin-1/2 operators on each site. The Hamiltonian is

$$\textbf{isotropic XY model:} \qquad H_{XY} = J\sum_{j=1}^{N}\left[S_j^x S_{j+1}^x + S_j^y S_{j+1}^y\right] - h\sum_{j=1}^{N} S_j^z,$$

where we consider the antiferromagnetic case $J > 0$, and in which we have also included an external field h in the z direction. For definiteness, we also adopt periodic boundary conditions, namely $S_{j+N}^a \equiv S_j^a$.

Show that under the Jordan-Wigner transformation, the XY model becomes equivalent to the fermionic Hamiltonian[1]

$$H_{XY,f} = \frac{J}{2}\sum_{j=1}^{N}\left[c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j\right] + h\sum_{j=1}^{N} c_j^\dagger c_j - h\frac{N}{2}.$$

Note that the original magnetic field h now takes the role of (minus) the chemical potential for the fermions.

[1] *Note: in this and the next sub-question, don't worry about the boundary conditions. To enforce periodic boundary conditions on spin operators, one should really take $c_{j+N} = (-1)^{1+\sum_{l=1}^{N}\hat{n}_l} c_j$. This just impacts precisely which lattice the momenta fall on, and does not matter for bulk properties.*

e) (5 pts) Using a Fourier transformation to momentum space (see **Useful Formulas** for suggested conventions), diagonalize the fermionic version of the XY model, obtaining the form

$$H_{XY,f} = \sum_k \xi_k c_k^\dagger c_k - h\frac{N}{2}.$$

Give the explicit form of ξ_k. What is the ground state for large fields $h > J$? What does the ground state become for fields $0 < h < J$?

f) (10 pts) Write the coherent state path integral representation for the partition function of the XY model, when it is kept in equilibrium at temperature T (here and after, remember the **Useful Formulas**). Compute this partition function by performing the (Grassmann) Gaussian integrals and the resulting Matsubara sum. Specializing to the limit of zero temperature, give an expression for the free energy per site $f = -\frac{T}{N}\ln Z$.

g) (5 pts) An interesting variation on the isotropic XY model is to introduce some anisotropy in the spin exhange terms, thereby getting the

anisotropic XY model: $$H_{XY} = J\sum_{j=1}^{N}\left[(1+\gamma)S_j^x S_{j+1}^x + (1-\gamma)S_j^y S_{j+1}^y\right] - h\sum_{j=1}^{N} S_j^z,$$

where the additional real parameter γ quantifies the degree of anisotropy.

Using the Jordan-Wigner transformation, show that the anisotropic XY model has the equivalent fermionic formulation

$$H_{XY,f} = \frac{J}{2}\sum_{j=1}^{N}\left[c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j + \gamma\left(c_{j+1}c_j + c_j^\dagger c_{j+1}^\dagger\right)\right] + h\sum_{j=1}^{N} c_j^\dagger c_j - h\frac{N}{2}.$$

Performing a Fourier transformation, show that this becomes of the form

$$H_{XY,f} = \sum_{k>0}\begin{pmatrix} c_k^\dagger & ic_{-k}\end{pmatrix}\begin{pmatrix} a_k & b_k \\ b_k & -a_k\end{pmatrix}\begin{pmatrix} c_k \\ -ic_{-k}^\dagger\end{pmatrix} + D$$

and give the expression for the functions a_k and b_k (also for the J, h and γ-dependent constant D if you want to impress me).

h)* (5 pts) Performing a Bogoliubov transformation (you can just use the **Useful Formulas** without rederivation) to 'rotated' operators $\tilde{c}$, show that the theory is diagonalized to the form

$$H_{XY,f} = \sum_k \tilde{\xi}_k \tilde{c}_k^\dagger \tilde{c}_k + \tilde{D}$$

in which $\tilde{D}$ is again a (J, h and γ-dependent) constant which you don't have to calculate (unless you feel like it). Give the explicit expression for the quasiparticle energy $\tilde{\xi}_k$. Give a summary description of the spectrum of excitations as a function of h/J and γ/J. What would you say is the main difference at low energies between the cases $\gamma = 0$ and $\gamma \neq 0$?

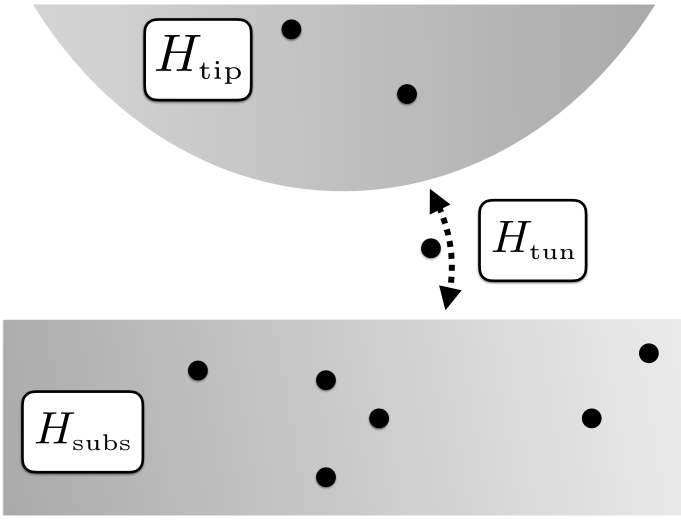

Figure 1: The Scanning Tunneling Microscope setup. A tip is put in close vicinity to a sample substrate. Within the tip (resp. substrate), electrons are described by Hamiltonian H_{tip} (resp. H_{subs}). When the tip is close enough to the substrate, a perturbative tunneling term H_{tun} allows electrons to hop between tip and substrate.

3. Tunneling spectroscopy (40 pts)

Scanning tunneling spectroscopy (STM) is a technique whereby sample surfaces can be probed with atomic resolution. It is based on the fact that particles (really: electrons) are able to quantum-mechanically 'tunnel' across the potential barrier between a tip and a substrate, with a tunneling rate which is greatly sensitive to details, in particular the tip-substrate distance. The basic setup is sketched in Fig. 1.

Both the tip and substrate can be viewed as 'bulk' systems supporting electrons (for simplicity: we forget about spin) with momentum-like quantum numbers k. We shall use the creation/annihilation operators $c_k, c_k^\dagger$ for electrons in the substrate and $d_k, d_k^\dagger$ in the tip, which obey canonical anticommutation relations

$$\left\{c_k, c_{k'}^\dagger\right\} = \delta_{kk'}, \qquad \left\{d_k, d_{k'}^\dagger\right\} = \delta_{kk'}$$

all other anticommutators vanishing. The substrate and tip Hamiltonians are given by (we put substrate and tip respectively at chemical potentials μ_c and μ_d)

$$H_{\text{subs}} - \mu_c N_{\text{subs}} = \sum_k (\varepsilon_{k,c} - \mu_c) c_k^\dagger c_k, \qquad H_{\text{tip}} - \mu_d N_{\text{tip}} = \sum_k (\varepsilon_{k,d} - \mu_d) d_k^\dagger d_k$$

with $N_{\text{subs}} = \sum_k c_k^\dagger c_k$ and $N_{\text{tip}} = \sum_k d_k^\dagger d_k$.

We now bring the tip and substrate in close proximity to each other, making tunneling possible. This is represented by the perturbation term

$$H_{\text{tun}} = \sum_{kk'} \left(t_{kk'} c_k^\dagger d_{k'} + t_{kk'}^* d_{k'}^\dagger c_k \right) \equiv T + T^\dagger$$

in which $t_{kk'}$ is the amplitude for tunneling from state k' in the tip to state k in the substrate. Our expectation is that if tip and substrate are at different chemical potentials (namely: at different voltages), there would be a current flowing from one to the other because of the tunneling term. This current can be defined for example as the rate of change of the charge in the tip:

$$I = \frac{d}{dt} N_{\text{tip}}.$$

a) (10 pts) Using the Heisenberg equation of motion $dA/dt = i[H, A]$, show that

$$I = J + J^\dagger, \qquad J \equiv i \sum_{kk'} t_{kk'} c_k^\dagger d_{k'}.$$

b) (10 pts) We now apply linear response theory. According to the Kubo formula, the current through the tip-substrate junction as a function of time is given by

$$\bar{I}(t) = \int_{-\infty}^{\infty} dt' \, \mathcal{C}_{ret}^{I,H_{\rm tun}}(t-t'), \qquad \mathcal{C}_{ret}^{I,H_{\rm tun}}(t-t') = -i\theta(t-t')\langle [I^I(t), H^I_{\rm tun}(t')] \rangle_0$$

in which the expectation value is the thermal, grand-canonical expectation value using the unperturbed theory $H_0 = H_{\rm subs} + H_{\rm tip}$ (don't forget to include the chemical potentials). Show that the current can be written in terms of the 'lesser' and 'greater' functions

$$\mathcal{C}^{c,>}_{\beta,\mu_c;k}(t_1 - t_2) = -i\langle c_k(t_1) c_k^\dagger(t_2)\rangle_0, \qquad \mathcal{C}^{c,<}_{\beta,\mu_c;k}(t_1 - t_2) = -i\zeta\langle c_k^\dagger(t_2) c_k(t_1)\rangle_0,$$
$$\mathcal{C}^{d,>}_{\beta,\mu_d;k}(t_1 - t_2) = -i\langle d_k(t_1) d_k^\dagger(t_2)\rangle_0, \qquad \mathcal{C}^{d,<}_{\beta,\mu_d;k}(t_1 - t_2) = -i\zeta\langle d_k^\dagger(t_2) d_k(t_1)\rangle_0,$$

(here of course, since we're dealing with fermions, $\zeta = -1$) as

$$\bar{I}(t) = 2 \text{ Re} \int_{-\infty}^{0} dt' \sum_{k_1 k_2} |t_{k_1 k_2}|^2 \left(\mathcal{C}^{c,<}_{\beta,\mu_c;k_1}(t') \mathcal{C}^{d,>}_{\beta,\mu_d;k_2}(-t') - \mathcal{C}^{c,>}_{\beta,\mu_c;k_1}(t') \mathcal{C}^{d,<}_{\beta,\mu_d;k_2}(-t') \right)$$

(note that this result becomes *time independent*: we are thus calculating a *static current*, which makes sense because our perturbation is also time-independent).

c) (10 pts) Fourier transforming the correlators according to

$$\mathcal{C}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \mathcal{C}(\omega),$$

making use of the Dirac identity in the form (P means the principal part under the integral sign[2])

$$\int_0^\infty dt e^{i\omega t} = \pi\delta(\omega) + iP\frac{1}{\omega},$$

and using the relations between the greater/lesser functions and the spectral function A

$$\mathcal{C}^{a,>}_{\beta,\mu_a;k}(\omega) = -i(1 - n_F(\omega - \mu_a;\beta)) A^a_{\beta,\mu_a;k}(\omega), \qquad \mathcal{C}^{a,<}_{\beta,\mu_a;k}(\omega) = i n_F(\omega - \mu_a;\beta) A^a_{\beta,\mu_a;k}(\omega), \qquad a = c, d,$$

show that the current is expressed as

$$\bar{I} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sum_{k_1 k_2} |t_{k_1 k_2}|^2 \left(n_F(\omega - \mu_c;\beta) - n_F(\omega - \mu_d;\beta) \right) A^c_{\beta,\mu_c;k_1}(\omega) A^d_{\beta,\mu_d;k_2}(\omega).$$

What happens to the current if we set $\mu_c = \mu_d$? Interpret this result.

d) (10 pts) We now make the assumption that this tip is metallic, so it's density of states is more or less constant, and that we can approximate

$$\sum_{k_2} |t_{k_1 k_2}|^2 A^d_{\beta,\mu_d;k_2}(\omega) \simeq 2\pi t_{k_1} \nu_{\rm tip}$$

as being ω-independent (t_{k_1} is some real positive function of k_1; $\nu_{\rm tip}$ is the density of states in the tip). Interpreting the chemical potential difference $\mu_c - \mu_d = V$ as a potential difference, show that the zero-temperature limit of the differential current (derivative of the current with respect to V at fixed μ_c) is a direct measure of the spectral function of the substrate's electrons,

$$\lim_{\beta\to\infty} \frac{d\bar{I}}{dV} = \nu_{\rm tip} \sum_{k_1} t_{k_1} A^c_{\beta,\mu_c;k_1}(\mu_c - V).$$

[2]Remember that you need the real part only, so this principal part term just drops out.

Useful Formulas

Trigonometric and hyperbolic functions

$$\sin(\theta_1+\theta_2)=\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2, \qquad \cos(\theta_1+\theta_2)=\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2,$$
$$\cos^2\theta+\sin^2\theta=1, \qquad \sin^2\theta=\frac{1}{2}(1-\cos 2\theta), \qquad \cos^2\theta=\frac{1}{2}(1+\cos 2\theta),$$
$$\sinh(\theta_1+\theta_2)=\sinh\theta_1\cosh\theta_2+\cosh\theta_1\sinh\theta_2, \quad \cosh(\theta_1+\theta_2)=\cosh\theta_1\cosh\theta_2+\sinh\theta_1\sinh\theta_2,$$
$$\cosh^2\theta-\sinh^2\theta=1, \qquad \sinh^2\theta=\frac{1}{2}(\cosh 2\theta-1), \qquad \cosh^2\theta=\frac{1}{2}(\cosh 2\theta+1).$$

Series expansions

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}, \qquad \cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}, \qquad \sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!},$$
$$(1+x)^\alpha=\sum_{n=0}^{\infty}\begin{pmatrix}\alpha\\ n\end{pmatrix}x^n=1+\alpha x+\frac{\alpha(\alpha-1)}{2}x^2+..., \qquad \ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$$

Bosonic occupation number states

$$[b,b^\dagger]=1, \qquad |n\rangle=\frac{1}{\sqrt{n!}}(b^\dagger)^n|0\rangle, \qquad b^\dagger|n\rangle=\sqrt{n+1}|n+1\rangle, \qquad b|n\rangle=\sqrt{n}|n-1\rangle.$$

Pauli spin matrices

$$\sigma^x=\begin{pmatrix}0&1\\1&0\end{pmatrix}, \qquad \sigma^y=\begin{pmatrix}0&-i\\i&0\end{pmatrix}, \qquad \sigma^z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \qquad \sigma^\pm=\frac{1}{2}(\sigma^x\pm i\sigma^y).$$

Spins on a lattice

$su(2)$ spin algebra (here, $i,j,k=x,y,z$ and m,n denote lattice sites and ε^{ijk} is the completely antisymmetric tensor with $\varepsilon^{ijk}=\pm1$ for $ijk=$ even/odd permutation of xyz, 0 otherwise).

$$\left[\hat{S}^i_m,\hat{S}^j_n\right]=i\delta_{mn}\varepsilon^{ijk}\hat{S}^k_n.$$

Spin raising and lowering operators: $\hat{S}^\pm_m=\hat{S}^x_m\pm i\hat{S}^y_m$ with

$$\left[\hat{S}^z_m,\hat{S}^\pm_n\right]=\pm\delta_{nm}\hat{S}^\pm_m, \qquad \left[\hat{S}^+_m,\hat{S}^-_n\right]=2\delta_{nm}\hat{S}^z_m.$$

For the $S=1/2$ case, one can use the representation $S^i=\sigma^i/2$, $i=x,y,z$.

Holstein-Primakoff transformation

$$\hat{S}^-_m=a^\dagger_m(2S-a^\dagger_m a_m)^{1/2}, \qquad \hat{S}^+_m=(2S-a^\dagger_m a_m)^{1/2}a_m, \qquad \hat{S}^z_m=S-a^\dagger_m a_m$$

where a_m, $a^\dagger_m$ are bosonic operators obeying the canonical algebra $[a_m,a^\dagger_n]=\delta_{mn}$ (other commutators vanish).

Fourier transformation

$$a_k=\frac{1}{\sqrt{N}}\sum_{m=1}^{N}e^{ikm}a_m, \quad a_m=\frac{1}{\sqrt{N}}\sum_{k\in BZ}e^{-ikm}a_k, \quad [a_k,a^\dagger_{k'}]_\zeta=\left\{\begin{array}{ll}a_k a^\dagger_{k'}-a^\dagger_{k'}a_k, & \text{bosons}\\ a_k a^\dagger_{k'}+a^\dagger_{k'}a_k, & \text{fermions}\end{array}\right.=\delta_{kk'}$$

Bogoliubov transformation

The matrix

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

(here for $a, b \in \mathbb{R}$) can be diagonalized by the unitary transformation

$$UHU^\dagger = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix}, \qquad U = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

where $\tan 2\theta = \frac{b}{a}$ and $\varepsilon = (a^2 + b^2)^{1/2}$.

Random walks

Diffusion equation:

$$\left(\frac{\partial}{\partial t} - D\,\nabla^2\right) P(\boldsymbol{r}, t) = 0.$$

In the scaling limit, for a d-dimensional hypercubic lattice, the diffusion constant D is related to the lattice spacing a, step time δt and dimension d by

$$D = \lim_{\substack{a\to 0\\ \delta t\to 0}} \frac{a^2}{2d\delta t}.$$

The probability per unit volume of being at position $\boldsymbol{r}_1$ at time t_1 given that one was at $\boldsymbol{r}_0$ and time t_0 is given by

$$\begin{aligned} p(\boldsymbol{r}_1, t_1|\boldsymbol{r}_0, t_0) \equiv \lim a^{-d} P_{\boldsymbol{r}_1,t_1|\boldsymbol{r}_0,t_0} &= \int_{-\infty}^{\infty} \frac{d^d\boldsymbol{k}}{(2\pi)^d} e^{-(t_1-t_0)D\boldsymbol{k}^2 + i\boldsymbol{k}\cdot(\boldsymbol{r}_1-\boldsymbol{r}_0)} \\ &= \frac{1}{[4\pi D(t_1-t_0)]^{\frac{d}{2}}} \exp\left[-\frac{|\boldsymbol{r}_1-\boldsymbol{r}_0|^2}{4D(t_1-t_0)}\right]. \end{aligned}$$

Coherent states (bosons: $\zeta = 1$, fermions: $\zeta = -1$)

$$|\phi\rangle \equiv \exp\left[\zeta \sum_i \phi_i a_i^\dagger\right] |0\rangle$$

$$a_i|\phi\rangle = \phi_i|\phi\rangle, \qquad a_i^\dagger|\phi\rangle = \zeta\partial_{\phi_i}|\phi\rangle, \qquad \langle\phi|a_i^\dagger = \langle\phi|\bar{\phi}_i, \qquad \langle\phi|a_i = \partial_{\bar{\phi}_i}\langle\phi| \qquad \forall i.$$

The norm of a coherent state is

$$\langle\phi|\phi\rangle = \exp\left[\sum_i \bar{\phi}_i\phi_i\right].$$

Coherent states form an (over)complete set of states:

$$\int \prod_i d(\bar{\phi}_i, \phi_i) e^{-\sum_i \bar{\phi}_i\phi_i} |\phi\rangle\langle\phi| = \mathbf{1}_{\mathcal{F}}$$

with $\mathbf{1}_{\mathcal{F}}$ the identity in Fock space. The measures are $d(\bar{\phi}_i, \phi_i) = \frac{d\bar{\phi}_i d\phi_i}{\pi}$ for bosons, $d(\bar{\phi}_i, \phi_i) = d\bar{\phi}_i d\phi_i$ for fermions.

Campbell-Baker-Hausdorff formula

The general identity called the Campbell-Baker-Hausdorff formula reads:

$$e^{-B}Ae^{B} = \sum_{n=0}^{\infty} \frac{1}{n!}[A,B]_n, \qquad \text{where } [A,B]_n = [[A,B]_{n-1},B], \quad [A,B]_0 \equiv A.$$

This can be specialized to some simpler particular cases. Let A and B be two quantum operators such that $[A,B]$ commutes with A and B. Then, the following identities hold:

$$e^{A+B} = e^{A}e^{B}e^{-\frac{1}{2}[A,B]}, \qquad [A, e^{\lambda B}] = \lambda [A,B] e^{\lambda B}.$$

Another useful one is:

$$\text{if } [A,B] = DB \text{ and } [A,D] = 0 = [B,D], \text{ then } f(A)B = Bf(A+D).$$

This then implies (under the same conditions) that

$$e^{A}Be^{-A} = Be^{D}.$$

Grassmann variables

$$\forall i,j, \quad \eta_i\eta_j = -\eta_j\eta_i, \qquad \int d\eta_i = 0, \qquad \int d\eta_i\eta_i = 1.$$

Coherent state path integral representation of the partition function

For a second-quantized Hamiltonian of the form

$$\hat{H}(a^\dagger, a) = \sum_{ij} h_{ij} a_i^\dagger a_j + \sum_{ijkl} V_{ijkl} a_i^\dagger a_j^\dagger a_k a_l,$$

the partition function is

$$\mathcal{Z} = \int \mathcal{D}(\bar{\psi}, \psi) e^{-S[\bar{\psi},\psi]}.$$

Here, we work directly in the Matsubara frequency (usually labeled by the index n, whose value runs over all integers) representation. The measure is defined as $\mathcal{D}(\bar{\psi},\psi) = \prod_i \prod_n d(\bar{\psi}_{in}, \psi_{in})$ and $d(\bar{\psi},\psi) \equiv \beta d\bar{\psi} d\psi$ for fermions and $d(\bar{\psi},\psi) \equiv \frac{1}{\pi\beta} d\bar{\psi} d\psi$ for bosons (see next subsection for the Gaussian integral). The effective action is

$$S[\bar{\psi},\psi] = \sum_{ij,n} \bar{\psi}_{in} \left[(-i\omega_n - \mu)\delta_{ij} + h_{ij}\right] \psi_{jn} + T \sum_{ijkl,\{n_i\}} V_{ijkl} \bar{\psi}_{in_1} \bar{\psi}_{jn_2} \psi_{kn_3} \psi_{ln_4} \delta_{n_1+n_2,n_3+n_4}.$$

Gaussian integration over bosonic/Grassmann variables

By definition, in the frequency representation of the action, we use

$$\int d(\bar{\psi},\psi) e^{-\bar{\psi}\varepsilon\psi} = (\beta\varepsilon)^{-\zeta}$$

with $\zeta = +1$ for bosons and -1 for fermions.

Wick's theorem (fermions)

The expectation value of a product of fermionic fields over a noninteracting theory is given by the sum over all pairings signed by the permutation order. For four fields,

$$\langle \bar{\psi}_a \bar{\psi}_b \psi_c \psi_d \rangle_0 = \langle \bar{\psi}_a \psi_d \rangle_0 \langle \bar{\psi}_b \psi_c \rangle_0 - \langle \bar{\psi}_a \psi_c \rangle_0 \langle \bar{\psi}_b \psi_d \rangle_0.$$

The first term is the Hartree term, the second is the Fock term.

Relations between Green's functions

retarded from imaginary-time: $\mathcal{C}^{ret}(\omega) = \mathcal{C}^{\tau}(i\omega_n)|_{i\omega_n \to \omega + i\eta}$

advanced from imaginary-time: $\mathcal{C}^{ret}(\omega) = \mathcal{C}^{\tau}(i\omega_n)|_{i\omega_n \to \omega - i\eta}$

Matsubara sums (fermions)

$$\sum_n \ln(\beta\left[-i\omega_n + \xi\right]) = \ln\left[1 + e^{-\beta\xi}\right],$$
$$T\sum_n \frac{1}{i\omega_n - \varepsilon_a + \mu} = \frac{1}{e^{\beta(\varepsilon_a - \mu)} + 1} \equiv n_F(\varepsilon_a, \mu).$$

Interaction representation

For the Hamiltonian $H = H_0 + H_I$ in which H_I represents the 'interaction' and H_0 the free (exactly-solvable) model, the interaction picture states and operators are related to the Schrödinger ones by

$$|\psi^I(t)\rangle = e^{iH_0 t}|\psi^S(t)\rangle, \qquad \mathcal{O}^I(t) = e^{iH_0 t}\mathcal{O}^S e^{-iH_0 t}.$$

Linear response theory: the Kubo formula

For the time-dependent Hamiltonian (in the Schrödinger picture)

$$H(t) = H_0 + F(t)\hat{P},$$

with initial condition that the system at $t \to -\infty$ is in state $|\psi_o\rangle$, the time-dependent expectation value of operator $\mathcal{O}$ is given in linear response by the Kubo formula

$$\bar{O}(t) = \langle\psi_0|\hat{O}|\psi_0\rangle + \int_{-\infty}^{\infty} dt' \mathcal{C}^{\hat{O},\hat{P}}_{ret,\psi_0}(t - t')F(t') + O(F^2)$$

in terms of the retarded correlation function (computed in state $|\psi_0\rangle$) between the perturbation and observable, this retarded function being defined (for a generic state $|\psi\rangle$) as

$$\mathcal{C}^{\hat{O},\hat{P}}_{ret,\psi}(t - t') \equiv -i\theta(t - t')\langle\psi|[\hat{O}^I(t), \hat{P}^I(t')]|\psi\rangle.$$

Stellar photon escape time: solution

The mean free path (distance a photon travels between scattering events) is given by a = collision time/speed of light $\simeq 3 \times 10^{-2}s$. The distance the photon needs to travel is the Sun's radius, which in units of the mean free path is $R_{\text{Sun}} \approx 7 \times 10^5$ km $\approx 2.3 \times 10^{10} r$. Since the mean distance D from the origin achieved in a random walk with N steps of length a is $\sim \sqrt{N}a$ (this scaling law is universal, in the sense that it does not depend on microscopic details of the walk), we thus get that to diffuse by a distance of R_{Sun}, we need $N \approx (2.3 \times 10^{10})^2 \approx 5 \times 10^{20}$ steps. Since each step takes a timescale of the collision time, the photon thus needs a time of about $N \times$ collision time $\approx 5 \times 10^{20} \times 10^{-10}s = 5 \times 10^{10}s \approx 1.4 \times 10^8$ hr $\approx 6 \times 10^6$ days ≈ 20000 years.

Of course we're not being precise here, so this is at best an order-of-magnitude estimate. A proper solution would take into account a number of facts, including:

- a scattering time (equivalently diffusion constant) dependent on position (distance from center);
- a prefactor to the $\sqrt{N}$ mean distance, coming from the 'geometry' of the scattering (namely: isotropic);
- compensating for the fact that this is really diffusion with boundary conditions in which the photon escapes once it reaches the surface, and cannot come back (in other words: we'd have to solve the diffusion equation in the presence of a boundary condition putting the probability of being at the boundary to zero).

Jordan-Wigner transformation; the XY model: solution

a)

$$\begin{aligned}
\left[S^z, S^+\right] &= \left[-c^\dagger c, c\right] = \left\{c^\dagger, c\right\} c = c = S^+,\\
\left[S^z, S^-\right] &= \left[-c^\dagger c, c^\dagger\right] = -c^\dagger \left\{c, c^\dagger\right\} = -c^\dagger = -S^-,\\
\left[S^+, S^-\right] &= \left[c, c^\dagger\right] = \left\{c, c^\dagger\right\} - 2c^\dagger c = 2S^z.
\end{aligned}$$

b) The commutations involving only site 1 operators are already checked. Since the factor $1-2\hat{n}_1$ in the expressions for the site 2 operators always commutes with the site 2 operators, we also have that the commutations of the form $\left[S_2^a, S_2^b\right]$ are obeyed.

The only nontrivial checks are between site 1 and 2 operators. We have that $[S_1^z, S_2^a] = 0$ since $\hat{n}_1$ commutes with itself and site 2 operators. Going further, $\left[S_1^\pm, S_2^z\right]$ also vanish. The only remaining ones are

$$\begin{aligned}
\left[S_1^+, S_2^+\right] &= \overbrace{c_1(1-2\hat{n}_1)}^{=-c_1} c_2 - (1-2\hat{n}_1)c_2 c_1 = -c_1 c_2 + \overbrace{(1-2\hat{n}_1)c_1}^{=c_1} c_2 = 0,\\
\left[S_1^+, S_2^-\right] &= \overbrace{c_1(1-2\hat{n}_1)}^{=-c_1} c_2^\dagger - (1-2\hat{n}_1)c_2^\dagger c_1 = -c_1 c_2^\dagger + \overbrace{(1-2\hat{n}_1)c_1}^{=c_1} c_2^\dagger = 0,\\
\left[S_1^-, S_2^+\right] &= \overbrace{c_1^\dagger(1-2\hat{n}_1)}^{=c_1^\dagger} c_2 - (1-2\hat{n}_1)c_2 c_1^\dagger = c_1^\dagger c_2 + \overbrace{(1-2\hat{n}_1)c_1^\dagger}^{=-c_1^\dagger} c_2 = 0,\\
\left[S_1^-, S_2^-\right] &= \overbrace{c_1^\dagger(1-2\hat{n}_1)}^{=c_1^\dagger} c_2^\dagger - (1-2\hat{n}_1)c_2^\dagger c_1^\dagger = c_1^\dagger c_2^\dagger + \overbrace{(1-2\hat{n}_1)c_1^\dagger}^{=-c_1^\dagger} c_2^\dagger = 0.
\end{aligned}$$

c) Let's consider $\left[S_j^+, S_{j'}^-\right]$. Begin by writing each term individually:

$$\begin{aligned}
S_j^+ S_{j'}^- &= \prod_{l=1}^{j-1}(1-2\hat{n}_l)\; c_j \prod_{l'=1}^{j'-1}(1-2\hat{n}_{l'})\; c_{j'}^\dagger,\\
S_{j'}^- S_j^+ &= \prod_{l'=1}^{j'-1}(1-2\hat{n}_{l'})\; c_{j'}^\dagger \prod_{l=1}^{j-1}(1-2\hat{n}_l)\; c_j.
\end{aligned}$$

Let's assume that $j < j'$ (the case $j > j'$ then follows by conjugation). Then, using the fact that $(1-2\hat{n}_l)^2 = 1$ for all $l < j$, we get

$$\begin{aligned}
S_j^+ S_{j'}^- &= \overbrace{c_j(1-2\hat{n}_j)}^{=-c_j} \prod_{l'=j+1}^{j'-1}(1-2\hat{n}_{l'})c_{j'}^\dagger = -\prod_{l'=j+1}^{j'-1}(1-2\hat{n}_{l'})c_j c_{j'}^\dagger,\\
S_{j'}^- S_j^+ &= \prod_{l'=j}^{j'-1}(1-2\hat{n}_{l'})\; c_{j'}^\dagger c_j = \prod_{l'=j+1}^{j'-1}(1-2\hat{n}_{l'})\; c_{j'}^\dagger\; \overbrace{(1-2\hat{n}_j)c_j}^{=c_j} = \prod_{l'=j+1}^{j'-1}(1-2\hat{n}_{l'})\; c_{j'}^\dagger c_j.
\end{aligned}$$

Therefore,

$$j < j': \qquad \left[S_j^+, S_{j'}^-\right] = -\prod_{l'=j+1}^{j'-1}(1-2\hat{n}_{l'})\; \left\{c_j, c_{j'}^\dagger\right\} = 0.$$

For $j = j'$, we get back to the original single-site case since all the Jordan-Wigner 'tail' factors square out.

Consider now $\left[S^z_j, S^+_{j'}\right]$. We have

$$S^z_j S^+_{j'} = (1-2\hat{n}_j)\prod_{l'=1}^{j'-1}(1-2\hat{n}_{l'})\ c_{j'}, \qquad S^+_{j'} S^z_j = \prod_{l'=1}^{j'-1}(1-2\hat{n}_{l'})\ c_{j'}(1-2\hat{n}_j).$$

Since

$$[1-2\hat{n}_j, c_{j'}] = -2\hat{n}_j c_{j'} + 2c_{j'}\hat{n}_j = 2\left\{c_{j'}, c^\dagger_j\right\} c_j = 2\delta_{jj'}c_j,$$

we thus obtain

$$\left[S^z_j, S^+_{j'}\right] = \prod_{l'=1}^{j'-1}(1-2\hat{n}_{l'})\ 2\delta_{jj'}c_j = 2\delta_{jj'}S^+_j$$

thus reproducing the correct spin operator commutation relation. The commutator $\left[S^z_j, S^-_{j'}\right]$ can be similarly computed, or most easily obtained by (minus) the conjugate of the last equation. We thus indeed find

$$\left[S^z_j, S^-_{j'}\right] = \prod_{l'=1}^{j'-1}(1-2\hat{n}_{l'})\ (-2)\delta_{jj'}c^\dagger_j = -2\delta_{jj'}S^-_j$$

so all canonical spin commutation relations have been accounted for correctly. Jordan-Wigner works!

d) Using

$$S^x_j S^x_{j+1} + S^y_j S^y_{j+1} = \frac{1}{2}\left(S^+_j S^-_{j+1} + S^-_j S^+_{j+1}\right)$$

and

$$S^+_j S^-_{j+1} = c_j(1-2\hat{n}_j)c^\dagger_{j+1} = -c_j c^\dagger_{j+1} = c^\dagger_{j+1}c_j, \qquad S^-_j S^+_{j+1} = c^\dagger_j(1-2\hat{n}_j)c_{j+1} = c^\dagger_j c_{j+1},$$

the bulk answer follows.

The following is not needed when answering during the exam: note that the boundary terms are

$$S^+_N S^-_1 = \prod_{l=1}^{N-1}(1-2\hat{n}_l)c_N c^\dagger_1 = \prod_{l=1}^{N}(1-2\hat{n}_l)c_N c^\dagger_1 = -\prod_{l=1}^{N-1}(1-2\hat{n}_l)c^\dagger_1 c_N, \quad S^-_N S^+_1 = -\prod_{l=1}^{N-1}(1-2\hat{n}_l)c^\dagger_N c_1.$$

The Hamiltonian is thus really

$$H_{XY,f} = \frac{J}{2}\sum_{j=1}^{N-1}\left[c^\dagger_j c_{j+1} + c^\dagger_{j+1}c_j\right] - \frac{J}{2}\left(\prod_{l=1}^{N}(1-2\hat{n}_l)\right)\ \left[c^\dagger_1 c_N + c^\dagger_N c_1\right] + h\sum_{j=1}^{N} c^\dagger_j c_j - h\frac{N}{2}.$$

Taking periodic boundary conditions on the spin operators thus translates into taking the boundary conditions

$$c_{N+1} = -\prod_{l=1}^{N}(1-2\hat{n}_l)c_1$$

on the fermions (namely: anti-periodic for even filling, periodic for odd).

e) The Fourier convention is $c_j = \frac{1}{\sqrt{N}}\sum_k e^{-ikj}c_k$ (the precise lattice for k depends on the boundary conditions, see answer to previous question; this is not important for the exam). The calculation goes straight into the answer with

$$\xi_k = J\cos k + h.$$

For fields $h > J$, $\xi_k > 0\forall k$ and thus the ground state is the fermionic vacuum. For values of field $0 < h < J$, ξ_k dips under zero for $|k| \in [\pi - \mathrm{acos}\frac{h}{J}, \pi]$. The ground state is thus a Fermi sea (interval) of filled states between $\pi - \mathrm{acos}\frac{h}{J}$ and $\pi + \mathrm{acos}\frac{h}{J}$ mod 2π (in other words: a Fermi sea of width $2\mathrm{acos}\frac{h}{J}$ centered on momentum π, momentum being 2π-periodic.

f) The Matsubara representation for the partition function is

$$Z = e^{\beta h \frac{N}{2}} \times \int \mathcal{D}(\bar{\psi}, \psi) e^{-S[\bar{\psi},\psi]}, \qquad S[\bar{\psi}, \psi] = \sum_k \sum_n \bar{\psi}_{k,n}[-i\omega_n + \xi_k]\psi_{k,n}.$$

The functional integral factorizes into $Z = e^{\beta h \frac{N}{2}} \times Z_{k,n}$, each term being of the form

$$Z_{k,n} = \int d(\bar{\psi}_{k,n}, \psi_{k,n}) e^{-\bar{\psi}_{k,n}[-i\omega_n+\xi_k]\psi_{k,n}} = \beta[-i\omega_n + \xi_k]$$

where in the last step we performed the Grassmann integration.

The free energy is thus (performing the Matsubara summation in the last equality)

$$f = -\frac{T}{N}\ln Z = -\frac{T}{N}\left(\beta h \frac{N}{2} + \sum_k \sum_n \beta[-i\omega_n + \xi_k]\right) = -\frac{h}{2} - \frac{T}{N}\sum_k \ln[1 + e^{-\beta\xi_k}].$$

In the $T \to 0$ limit, we get

$$f \to -\frac{h}{2} + \frac{1}{N}\sum_{k:\xi_k<0} \xi_k.$$

Though this was not asked in the question, for your information, this can be evaluated in the thermodynamic limit, using the fact that the Fermi sea is filled between $\pi - \text{acos}\frac{h}{J}$ and $\pi + \text{acos}\frac{h}{J}$ (exploiting 2π-periodicity of the Brillouin zone):

$$f \to -\frac{h}{2} + \int_{\pi-\text{acos}\frac{h}{J}}^{\pi+\text{acos}\frac{h}{J}} dk(J\cos k + h) = -\frac{h}{2} + J\sin k\Big|_{\pi-\text{acos}\frac{h}{J}}^{\pi+\text{acos}\frac{h}{J}} + 2h\ \text{acos}\frac{h}{J}.$$

But $\sin(\pi \pm \text{acos}\frac{h}{J}) = \mp \sin\text{acos}\frac{h}{J} = \sqrt{1 - \left(\frac{h}{J}\right)^2}$ so

$$f \to h\left(2\text{acos}\frac{h}{J} - \frac{1}{2}\right) - 2J\sqrt{1 - \left(\frac{h}{J}\right)^2}.$$

For $h = J$, we get $f = -\frac{h}{2} = -\frac{J}{2}$ as expected (the Fermi sea is empty, the magnet is completely polarized along $+\hat{z}$). For smaller values of h we start filling up the Fermi sea, until we reach $h = -J$, at which time it's completely filled. We then get $f = -\frac{h}{2} = \frac{J}{2}$, as per a system completely polarized but now in the $-\hat{z}$ direction.

g) For the anisotropic case, we only need to add the $\propto \gamma$ terms:

$$\begin{aligned} S_j^x S_{j+1}^x - S_j^y S_{j+1}^y &= \frac{1}{2}\left(S_j^+ S_{j+1}^+ + S_j^- S_{j+1}^-\right) \\ &= \frac{1}{2}\left(c_j(1-2\hat{n}_j)c_{j+1} + c_j^\dagger(1-2\hat{n}_j)c_{j+1}^\dagger\right) = \frac{1}{2}\left(-c^j c_{j+1} + c_j^\dagger c_{j+1}^\dagger\right) \end{aligned}$$

giving the first part of the answer. Under a Fourier transform, we get

$$\sum_j c_{j+1}c_j = \frac{1}{N}\sum_{k,k'} c_{k'}c_k e^{-ik'} \overbrace{\sum_j e^{-i(k+k')j}}^{=N\delta_{k+k',0}} = \sum_k c_{-k}c_k e^{ik} = i\sum_{k>0} c_{-k}c_k \sin k$$

where in the last equality we have used symmetry (for periodic boundary conditions, the $k = 0$ term is zero anyway). Adding this (and its conjugate) to the Hamiltonian gives

$$
\begin{aligned}
H_{XY,f} &= \sum_{k>0} \left\{ (J\cos k + h)\ [c_k^\dagger c_k + \overbrace{c_{-k}^\dagger c_{-k}}^{=-c_k c_{-k}^\dagger + 1}\ ] + i\gamma \sin k\ [c_{-k}c_k - c_k^\dagger c_{-k}^\dagger] \right\} - h\frac{N}{2} \\
&= \sum_{k>0} \left[a_k (c_k^\dagger c_k - c_{-k} c_{-k}^\dagger) + b_k (i c_{-k} c_k - i c_k^\dagger c_{-k}^\dagger) \right] + \sum_{k>0} (J\cos k + h) - h\frac{N}{2} \\
&= \sum_{k>0} \begin{pmatrix} c_k^\dagger & i c_{-k} \end{pmatrix} \begin{pmatrix} a_k & b_k \\ b_k & -a_k \end{pmatrix} \begin{pmatrix} c_k \\ -i c_{-k}^\dagger \end{pmatrix} + \sum_{k>0} (J\cos k + h) - h\frac{N}{2}
\end{aligned}
$$

with

$$
a_k \equiv J\cos k + h, \quad b_k \equiv J\gamma \sin k.
$$

h)* The Bogoliubov transformation is as per the Useful Formula, with

$$
\begin{pmatrix} \tilde{c}_k \\ -i\tilde{c}_{-k}^\dagger \end{pmatrix} = U_k \begin{pmatrix} c_k \\ -i c_{-k}^\dagger \end{pmatrix}, \qquad U_k = \begin{pmatrix} \cos\theta_k & \sin\theta_k \\ \sin\theta_k & -\cos\theta_k \end{pmatrix}
$$

where $\tan 2\theta_k = \frac{b_k}{a_k}$. The Hamiltonian matrix becomes

$$
UHU^\dagger = \begin{pmatrix} \xi_k & 0 \\ 0 & -\xi_k \end{pmatrix},
$$

where $\xi_k = (a_k^2 + b_k^2)^{1/2} = J\left[(\cos k + h/J) + \gamma^2 \sin^2 k\right]^{1/2}$. In total, we thus have

$$
\begin{aligned}
H_{XY,f} &= \sum_{k>0} \tilde{\xi}_k \left[\tilde{c}_k^\dagger \tilde{c}_k - \tilde{c}_{-k} \tilde{c}_{-k}^\dagger \right] + \sum_{k>0} (J\cos k + h) - h\frac{N}{2} \\
&= \sum_k \xi_k \tilde{c}_k^\dagger \tilde{c}_k + \sum_{k>0} (J\cos k + h - \xi_k) - h\frac{N}{2}.
\end{aligned}
$$

The single-particle spectrum ξ_k is strictly positive for any k when $\gamma \neq 0$, irrespective of the field h. The ground state of the theory is thus always the vacuum for the $\tilde{c}$ operators. For large h, we have a gapped spin-wave-like spectrum $\xi_k \approx h + J\cos k + ...$ For a given $\gamma \neq 0$, the spectrum is always gapped, meaning that there are no low-energy excitations (this being the main difference with the isotropic $\gamma = 0$ case). The XY model is in this sense the spin equivalent of a conventional superconductor.

Tunneling spectroscopy: solution

a)

$$I = i[H, N_{\text{tip}}] = i[H_{\text{tun}}, N_{\text{tip}}] = \sum_{k_1,k_2,k_3} it_{k_1k_2}[c^\dagger_{k_1} d_{k_2}, d^\dagger_{k_3} d_{k_3}] + \text{h.c.}$$
$$= \sum_{k_1,k_2,k_3} it_{k_1k_2} c^\dagger_{k_1}\{d_{k_2}, d^\dagger_{k_3}\} d_{k_3} + \text{h.c.} = i\sum_{kk'} t_{kk'} c^\dagger_k d_{k'} + \text{h.c.}.$$

b)

$$\mathcal{C}^{I,H_{\text{tun}}}_{ret}(t-t') = -i\theta(t-t')\langle\left[I^I(t), H_{\text{tun}}{}t^I(t')\right]\rangle = -i\theta(t-t')\left(\langle\left[J^I(t),(T^\dagger)^I(t')\right]\rangle + \langle\left[(J^\dagger)^I(t), T^I(t')\right]\rangle\right)$$

Looking at the first term (the second is its hermitian conjugate), using Wick's theorem and assuming that the correlators are purely diagonal in their indices,

$$\theta(t-t')\sum_{k_1k_2k_3k_4} t_{k_1k_2}t^*_{k_3k_4}\langle\left[c^\dagger_{k_1}(t)d_{k_2}(t), d^\dagger_{k_4}(t')c_{k_3}(t')\right]\rangle$$
$$= \theta(t-t')\sum_{k_1k_2k_3k_4} t_{k_1k_2}t^*_{k_3k_4}\left(\langle c^\dagger_{k_1}(t)c_{k_3}(t')\rangle_{\beta,\mu_c}\langle d_{k_2}(t)d^\dagger_{k_4}(t')\rangle_{\beta,\mu_d} - \langle c_{k_3}(t')c^\dagger_{k_1}(t)\rangle_{\beta,\mu_c}\langle d^\dagger_{k_4}(t')d_{k_2}(t)\rangle_{\beta,\mu_d}\right)$$
$$= \theta(t-t')\sum_{k_1k_2}|t_{k_1k_2}|^2\left(\langle c^\dagger_{k_1}(t)c_{k_1}(t')\rangle_{\beta,\mu_c}\langle d_{k_2}(t)d^\dagger_{k_2}(t')\rangle_{\beta,\mu_d} - \langle c_{k_1}(t')c^\dagger_{k_1}(t)\rangle_{\beta,\mu_c}\langle d^\dagger_{k_2}(t')d_{k_2}(t)\rangle_{\beta,\mu_d}\right)$$

By using the definition of the 'greater' and 'lesser' functions in the question, we immediately get that the first term is

$$\theta(t-t')\sum_{k_1k_2}|t_{k_1k_2}|^2\left(\mathcal{C}^{c,<}_{\beta,\mu_c;k_1}(t'-t)\mathcal{C}^{d,>}_{\beta,\mu_d;k_2}(t-t') - \mathcal{C}^{c,>}_{\beta,\mu_c;k_1}(t'-t)\mathcal{C}^{d,<}_{\beta,\mu_d;k_2}(t-t')\right).$$

Putting this in the Kubo formula (shifting the time integration parameter t' by t for convenience) then gives the answer.

c) This is straightforward. The principal part integral can be dropped since we only need the real part.

d) In the low temperature limit, we have that $\lim_{\beta\to\infty}\frac{d}{d\omega}n_F(\omega;\beta) = -\delta(\omega)$. This readily gives the answer.