Random walker on square lattice in d dim$^{\text{ns}}$

Lattice: $\quad \mathcal{L}_a \equiv \{\underset{\sim}{n}\} \qquad \underset{\sim}{n} = a \sum_{\mu=1}^{d} n^{\mu}\, \underset{\sim}{n}_{\mu}$

lattice spacing $\quad \in \mathbb{Z} \quad$ unit vector in dir$^{\text{n}}$ μ

Rules:

1: at each δt, the walker takes $\underline{1}$ step

2: the direction is chosen uniformly randomly among all $2d$ possibilities.

Q: what is the conditional proba

$P_{\underset{\sim}{n}_1, t_1 | \underset{\sim}{n}_0, t_0}$ that walker is at $\underset{\sim}{n}_1$ at time t_1, given that it was at $\underset{\sim}{n}_0$ at t_0.

$$P_{\underset{\sim}{n}_1 - \underset{\sim}{n}_0, t_1 | \underset{\sim}{n}_0 - \underset{\sim}{n}_0, t_0} \qquad P_{\underset{\sim}{n}_1, t_1 + t | \underset{\sim}{n}_0, t_0 + t}$$

can choose $P_{\underset{\sim}{n}_1, t_0 | \underset{\sim}{n}_0, t_0} = \delta_{\underset{\sim}{n}_1, \underset{\sim}{n}_0}$

Simple properties: • $0 \leq P_{\underset{\sim}{n}_1, t_1 | \underset{\sim}{n}_0, t_0} \leq 1$

"Sum rule": $$\sum_{\underset{\sim}{n} \in \mathcal{L}_a} P_{\underset{\sim}{n}, t_1 | \underset{\sim}{n}_0, t_0} = 1$$

If $|\underset{\sim}{n}_1 - \underset{\sim}{n}_0| > a \frac{t_1 - t_0}{\delta t}$, $P = 0$.

"Time evolution" of P.

$$P_{\underset{\sim}{r}, t_1 + \delta t \mid \underset{\sim}{r}_0, t_0} = \frac{1}{c} \sum_{\underset{\sim}{r}' \text{ nn } \underset{\sim}{r}} P_{\underset{\sim}{r}', t_1 \mid \underset{\sim}{r}_0, t_0}$$

$\mathcal{L}_a$

"coordination no."
$\equiv$ # neighbours

Particular case of hypercubic: $c = 2d$ so

$$P_{\underset{\sim}{r}, t_1 + \delta t \mid \underset{\sim}{r}_0, t_0} = \frac{1}{2d} \sum_{\sigma = \pm} \sum_{\mu=1}^{d} P_{\underset{\sim}{r} + a\sigma \underset{\sim}{n}_\mu, t_1 \mid \underset{\sim}{r}_0, t_0}$$

Let's introduce a discrete difference op:

$$\nabla_a^2 f_{\underset{\sim}{n}} \equiv \frac{1}{a^2}\sum_{\mu=1}^{d}\left[f_{\underset{\sim}{n}+a\underset{\sim}{m}_\mu} + f_{\underset{\sim}{n}-a\underset{\sim}{m}_\mu} - 2f_{\underset{\sim}{n}}\right]$$

seen f^{n} defined on $\mathcal{L}_a$

This gives a nice form for our time evol^{n}:

$$\boxed{P_{\underset{\sim}{n}_1,t_1+\delta t|\underset{\sim}{n}_0,t_0} - P_{\underset{\sim}{n}_1,t_1|\underset{\sim}{n}_0,t_0} = \frac{a^2}{2d}\nabla_a^2 P_{\underset{\sim}{n}_1,t_1|\underset{\sim}{n}_0,t_0}}$$

Going to the continuum in space & time,

we can write $\frac{\partial}{\partial t} P(\underset{\sim}{r}, t) = D \nabla^2 P(\underset{\sim}{r}, t)$

"scaling limit":

$a \to 0$, $\delta t \to 0$

s.t. $\frac{a^2}{\delta t}$ finite

$\lim_{\substack{a \to 0 \\ \delta t \to 0}} \frac{a^2}{2d\,\delta t}$

D: diffusion constant

Diffusion equation

Exact solution: from Fourier series.

Conventions:
$$f_{\mathbf{n}} = a^d \int_{-\pi/a}^{\pi/a} \frac{d^d \mathbf{k}}{(2\pi)^d} e^{i \mathbf{k} \cdot \mathbf{n}} f(\mathbf{k})$$

$$f(\mathbf{k}) = \sum_{\mathbf{n} \in \mathcal{L}_a} e^{-i \mathbf{k} \cdot \mathbf{n}} f_{\mathbf{n}}$$

FT of initial cond^n:

$$P_{\mathbf{n}, t_0 | \mathbf{n}_0, t_0} = \delta_{\mathbf{n}, \mathbf{n}_0} \longrightarrow P_{\mathbf{k}, t_0 | \mathbf{n}_0, t_0} = e^{-i \mathbf{k} \cdot \mathbf{n}_0}$$

FT of time evol^n eq^n:

$$P_{\underset{\sim}{n}_1, t_1+\delta t \mid \underset{\sim}{n}_0, t_0} = \frac{1}{2d} \sum_{\sigma=\pm} \sum_{\mu=1}^{d} P_{\underset{\sim}{n}_1 + a\sigma \underset{\sim}{n}_\mu, t_1 \mid \underset{\sim}{n}_0, t_0}$$

LHS: $$\sum_{\underset{\sim}{n}_1 \in \mathcal{L}} e^{-i \underset{\sim}{k} \cdot \underset{\sim}{n}_1} P_{\underset{\sim}{n}_1, t_1+\delta t} = P_{\underset{\sim}{k}, t_1+\delta t}$$

RHS: $$\frac{1}{2d} \sum_{\sigma=\pm} \sum_{\mu=1}^{d} \sum_{\underset{\sim}{n}_1 \in \mathcal{L}_a} e^{-i \underset{\sim}{k} \cdot \underset{\sim}{n}_1} P_{\underset{\sim}{n}_1 + a\sigma n_\mu, t_1}$$

$$= \frac{1}{2d} \sum_{\sigma} \sum_{\mu} \sum_{\underset{\sim}{n}_1 \in \mathcal{L}_a} e^{+i k_\mu a\sigma} e^{-i \underset{\sim}{k} \cdot \underset{\sim}{n}_1} P_{\underset{\sim}{n}_1, t_1} = \frac{1}{d} \sum_{\mu} \cos k_\mu a \; P_{\underset{\sim}{k}, t_1}$$

So $$P_{\underset{\sim}{k}, t_1+\delta t} = \left[\frac{1}{d}\sum_{\mu} \cos k_{\mu} a\right] P_{\underset{\sim}{k}, t_1}$$

So general sol$^{\underline{n}}$: $$P_{\underset{\sim}{k}, t_1+n\delta t} = \left[\frac{1}{d}\sum_{\mu} \cos k_{\mu} a\right]^{n} P_{\underset{\sim}{k}, t_1}$$

Last step: FT back to real space:

$$P_{\underset{\sim}{r}, t_1 | \underset{\sim}{r}_0, t_0} = a^d \int_{-\pi/a}^{\pi/a} \frac{d^d \underset{\sim}{k}}{(2\pi)^d} e^{i \underset{\sim}{k} \cdot (\underset{\sim}{r}_1 - \underset{\sim}{r}_0)} \left[\frac{1}{d}\sum_{\mu} \cos k_{\mu} a\right]^{\frac{t_1 - t_0}{\delta t}}$$

Continuum limit of sol^n: only region $|\underset{\sim}{k}| \simeq 0$ stays:

$$\left[\frac{1}{d}\sum_\mu \cos k_\mu a\right]^{\frac{t_1-t_0}{\delta t}} = \left[\frac{1}{d}\sum_\mu\left(1-\frac{k_\mu^2 a^2}{2}+\dots\right)\right]^{\frac{t_1-t_0}{\delta t}}$$

$$\lim\left[1+\frac{a}{n}\right]^n = e^a$$

$$= \left[1-\frac{a^2}{2d}\underbrace{\sum_\mu k_\mu^2}_{\underset{\sim}{k}^2}+\dots\right]^{\frac{t_1-t_0}{\delta t}} \to e^{-(t_1-t_0)\underbrace{\frac{a^2}{2d\,\delta t}}_{D}\underset{\sim}{k}^2}$$

= prob per unit volume

So

$$p(\underset{\sim}{x}_1,t_1|\underset{\sim}{x}_0,t_0) \equiv \lim_{\substack{a\to 0\\ \delta t\to 0}} \frac{P_{\underset{\sim}{x}_1,t_1}}{a^d} = \int_{-\infty}^{\infty}\frac{d^d\underset{\sim}{k}}{(2\pi)^d}\, e^{-(t_1-t_0)D\underset{\sim}{k}^2+i\underset{\sim}{k}\cdot(\underset{\sim}{x}_1-\underset{\sim}{x}_0)}$$

Simple Gaussian integral, result is

$$\boxed{P(\underset{\sim}{r}_1, t_1 | \underset{\sim}{r}_0, t_0) = \frac{1}{[4\pi D(t_1 - t_0)]^{\frac{d}{2}}} \exp\left\{-\frac{|\underset{\sim}{r}_1 - \underset{\sim}{r}_0|^2}{4D(t_1 - t_0)}\right\}}$$

Summary: Diffusion $\left(\frac{\partial}{\partial t} - D\nabla^2\right) P(\underset{\sim}{r}, t | \underset{\sim}{r}_0, t_0) = 0$

Normalizⁿ $\int d^d \underset{\sim}{r}\, p(\underset{\sim}{r}, t | \underset{\sim}{r}_0, t_0) = 1$

Composition: $\int d^d \underset{\sim}{r}_1\, P(\underset{\sim}{r}_2, t_2 | \underset{\sim}{r}_1, t_1)\, P(\underset{\sim}{r}_1, t_1 | \underset{\sim}{r}_0, t_0) = P(\underset{\sim}{r}_2, t_2 | \underset{\sim}{r}_0, t_0)$

Further Q^{n}: how long does a walker spend on a particular pt? (during an ∞ long walk)

Answer: $\displaystyle\sum_{n=0}^{\infty} P_{\underset{\sim}{r},\, t_0+n\delta t \,|\, \underset{\sim}{r}_0,\, t_0} = a^d \int_{-\pi/a}^{\pi/a} \frac{d^d k}{(2\pi)^d} \frac{e^{i\underset{\sim}{k}\cdot(\underset{\sim}{r}-\underset{\sim}{r}_0)}}{1-\left[\frac{1}{d}\sum_{\mu}\cos k_{\mu}a\right]} \equiv \mathcal{G}_{\underset{\sim}{r}-\underset{\sim}{r}_0}$

Sum this using $\displaystyle\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$

$\mathcal{G}$ obeys $\displaystyle -\nabla_a^2 \mathcal{G}_{\underset{\sim}{r}-\underset{\sim}{r}_0} = \frac{2d}{a^2}\,\delta_{\underset{\sim}{r},\underset{\sim}{r}_0}$ (see notes)

Check more carefully!

$$\int d^d \underset{\sim}{k} \frac{e^{i \underset{\sim}{k} \cdot \underset{\sim}{l}}}{1 - \frac{1}{d}\sum_{\mu=1}^{d} \cos k_\mu a}$$

$d=3$: near $k \to 0$,

$$\sim \underbrace{\int dk\, \cancel{k^2}}_{\text{spherical coord. in 3d}} \times \frac{1}{\cancel{N}\left(\cancel{1}-\cancel{1}+k^2\right)}$$

$\cancel{k^2}$

$d=2$: $\sim \int \frac{dk\, k}{k^2} \to$??? log divergence at $k \to 0$

$d=1$: $\sim \int \frac{dk}{k^2}$!!! $\frac{1}{k}$ divergence!

First simple "solution": subtraction

Consider the "subtracted" Green's f^{n}

$\sim k^2$ at small k

$$\mathcal{G}^{S}_{\underset{\sim}{n}_1-\underset{\sim}{n}_0} = \mathcal{G}_{\underset{\sim}{n}_1-\underset{\sim}{n}_0} - \mathcal{G}_{\underset{\sim}{0}} \propto \int d^d \underset{\sim}{k}\ \frac{e^{i\underset{\sim}{k}a\cdot(\underset{\sim}{n}_1-\underset{\sim}{n}_0)} - 1}{1-\frac{1}{d}\sum_{\mu}\cos k_{\mu}a}$$

For explicit results in $d=1,(2,3,)$ see notes p.13

2^{nd} "solution": regularization

Idea: "exhaustible" walker

rule 3: at each time step, walker quits the game with probability $\eta > 0$.

Simple sol$^{\underline{n}}$: add a decay factor $(1-\eta)^n$ to previous sol$^{\underline{n}}$

Divergence gone: $\mathcal{G} \sim \int d^d k \; \dfrac{e^{ikr}}{\frac{1}{1-\eta} - \left[\frac{1}{d}\sum \cos k_a a\right]}$ See notes pp. 13-15

Path integral repres^{n}

$$P_{\underset{\sim}{r}_1, t_1 | \underset{\sim}{r}_0, t_0} = \frac{\#\text{ paths linking } \underset{\sim}{r}_0 \text{ to } \underset{\sim}{r}_1 \text{ with } \frac{t_1 - t_0}{\delta t} \text{ steps}}{\text{total } \# \text{ of paths with } \frac{t_1 - t_0}{\delta t} \text{ steps}}$$

In scaling limit.

We know $p(\underset{\sim}{r}_1, t_1 | \underset{\sim}{r}_0, t_0)$. Using composition: N steps

$$p(\underset{\sim}{r}_f, t_f | \underset{\sim}{r}_i, t_i) = \int \prod_{n=1}^{N-1} d^d \underset{\sim}{r}_n \; P(\underset{\sim}{r}_f, t_f | \underset{\sim}{r}_{N-1}, t_{N-1}) P(\underset{\sim}{r}_{N-1}, t_{N-1} | \underset{\sim}{r}_{N-2}, t_{N-2}) \times$$

$$\ldots \ldots, P(\underset{\sim}{r}_1, t_1 | \underset{\sim}{r}_i, t_i)$$

Taking N big, time steps $\frac{t_f - t_i}{N}$ become small.

Use $P\left(\underset{\sim}{r}_{m+1}, t_{m+1} \middle| \underset{\sim}{r}_m, t_m\right) \xrightarrow[N\to\infty]{} \frac{1}{\left[4\pi D \Delta t\right]^{\frac{d}{2}}} \exp\left\{-\frac{\Delta t}{4D}\left|\frac{\Delta r(t_m)}{\Delta t}\right|^2\right\}$

$\frac{dr}{dt}$

$t_{m+1} = t_m + \Delta t \qquad \Delta t \equiv \frac{t_f - t_i}{N}$

Thus: get path integral repres$\hat{~}$

$$P\left(\underset{\sim}{r}_f, t_f \middle| \underset{\sim}{r}_i, t_i\right) = \int \mathcal{D}\underset{\sim}{r}(t)\, e^{-\frac{1}{4D}\int_{t_i}^{t_f} dt \left(\frac{d\underset{\sim}{r}}{dt}\right)^2}$$

$\underset{\sim}{r}(0) = \underset{\sim}{r}_i, \ \underset{\sim}{r}(t) = \underset{\sim}{r}_f$

$$\equiv \lim_{N\to\infty}\left[\frac{N}{4\pi D(t_f - t_i)}\right]^{\frac{Nd}{2}} \int \prod_{n=1}^{N-1} d\underset{\sim}{r}_n$$