The Ising model

Lattice discrete degrees of freedom with 2 values: $S_i = \pm 1$

Spins: coupled pairwise in the Hamiltonian

$$H = \frac{1}{2} \sum_{i,j} J_{ij} S_i S_j - B \sum_i S_i$$

B: external field

Restrict to nearest-neighbour coupling.

Objective: compute the part^n f^n $Z = \sum_{\{S_i\}} e^{-\beta H(\{S_i\})}$

Solution of 1-d Ising model

Set $J \equiv -\varepsilon$ $\quad H = \sum_{i=0}^{N-1} H_{S_i S_{i+1}}$ $\quad H_{S_i S_{i+1}} = -\varepsilon S_i S_{i+1} + \frac{B}{2}(S_i + S_{i+1})$

Use periodic BCs: $S_N \equiv S_0$

$$Z = \sum_{\{S_i\}} \prod_{i=0}^{N-1} \exp\left\{-\beta H_{S_i S_{i+1}}\right\} = \mathrm{Tr}\, T^N$$

↑ transfer matrix

$$T_{SS'} \equiv e^{-\beta H_{SS'}} = \begin{pmatrix} e^{\beta(\varepsilon + B)} & e^{-\beta\varepsilon} \\ e^{-\beta\varepsilon} & e^{\beta(\varepsilon - B)} \end{pmatrix}$$

$$(T^2)_{ab} = \sum_c T_{ac} T_{cb}$$

To compute Z: diagonalize T, get eigenvalues λ_0, λ_1

$Z = \mathrm{Tr}\, T^N = \lambda_0^N + \lambda_1^N$ Choose labelling s.t. $|\lambda_0| > |\lambda_1|$

Then the free energy density $f = -\frac{T}{N} \ln Z$

becomes $f = -T \ln \lambda_0$

Simple calculation: $\lambda_0 = [\text{eq}^{\text{n}}\ 99]$

See notes for details. In particular:

$\langle s \rangle = \sinh \beta B / \sqrt{\dots}$ $\rightarrow$ no magnet$^{\text{n}}$ at $B = 0$

High-temperature expansion

Let's set $B=0$. $Z = \sum_{\{S_i\}} \prod_{\langle i,j \rangle} e^{\varepsilon\beta S_i S_j}$

lattice index in d dim^{ns}

nearest neighbours

Simple identity:

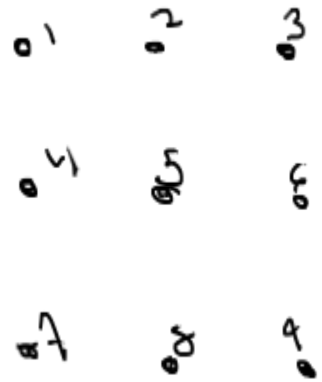
$$e^{\pm A} = \cosh A \pm \sinh A$$

$$= \cosh A \left[1 \pm \tanh A\right]$$ so $$e^{\varepsilon\beta S_i S_j} = \cosh \varepsilon\beta \left[1 + S_i S_j \underbrace{v}_{\tanh \varepsilon\beta}\right]$$

Can expand Z as a polynomial in v:

$$Z = \left[\cosh \varepsilon\beta\right]^{NZ/2} \sum_{\{S_i\}} \prod_{\langle i,j \rangle} \left[1 + S_i S_j v\right]$$

Pictorial repres^{n}: $S_i S_j$ $\longleftrightarrow$ 



 $\longleftrightarrow$ $S_4 S_7 S_5 S_6 v^2$

$$\sum_{\{S_i\}} = \sum_{S_1=\pm1}\sum_{S_2=\pm1}\sum_{S_3=\pm1}$$

 $\longleftrightarrow$ $S_1 \overset{1}{\cancel{S_4 S_4}} S_5 S_5 S_2 v^3$

 $\longleftrightarrow$ $\overset{1}{\cancel{S_1 S_4 S_4 S_5 S_5 S_2 S_2 S_1}} v^4$

Thus:

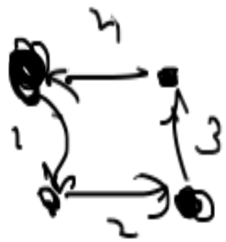
$N z/2 \to \infty$

$$Z = [\cosh \beta \varepsilon]^{\frac{Nz}{2}} \, 2^N \sum_{l=0}^{\infty} g(l) \, v^l$$

# of closed loops of length l

Setting things up:

- a <u>closed path</u> of l links: an arrowed path, from one point back to the same pt.



A closed path is called <u>connected</u> if it is made of a single body of links.

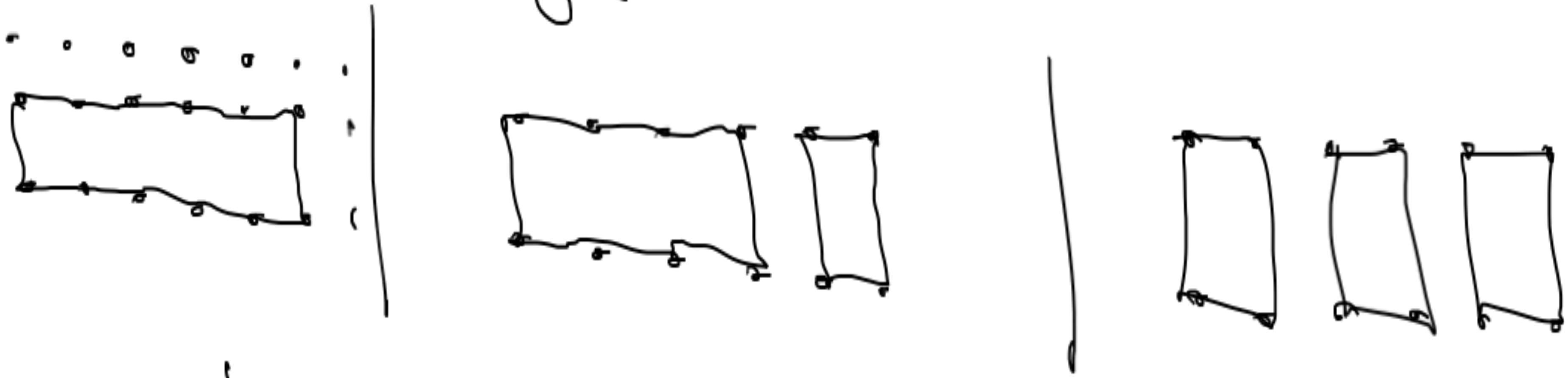
Let's call $h(\ell)$ the # of closed, connected paths of length ℓ.

Define a handy f^n: $D(\ell) \equiv \frac{1}{2\ell} h(\ell)$

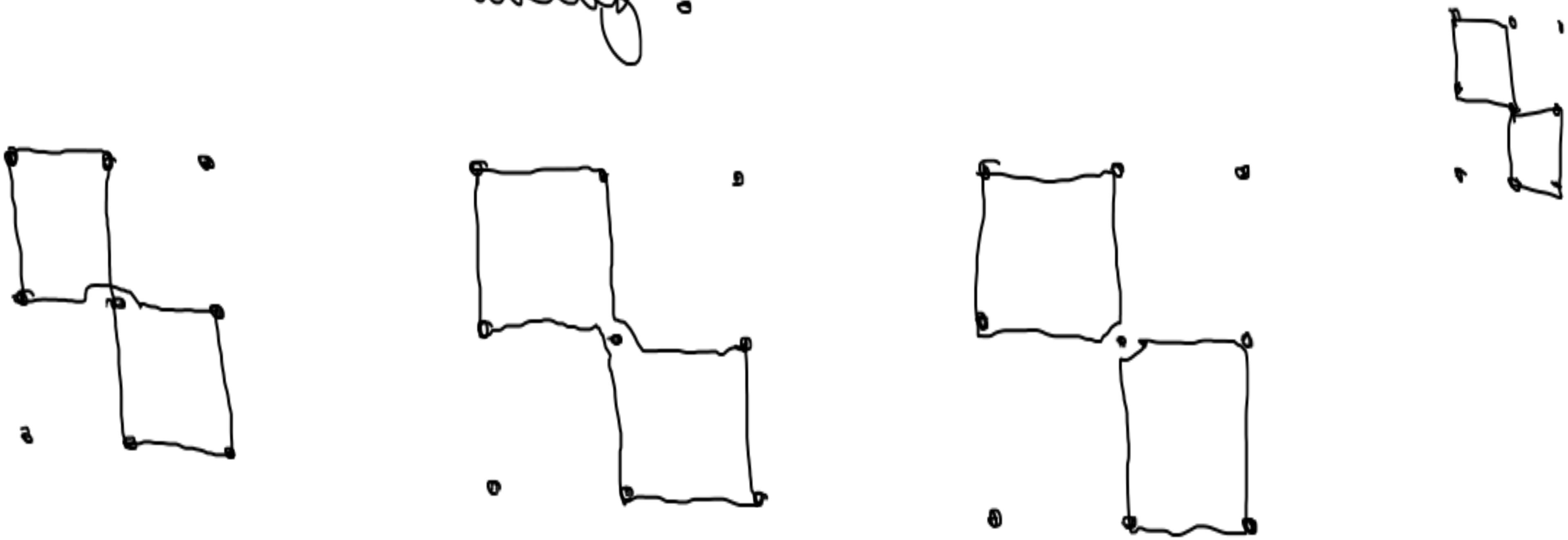
1st guess: total # of loops of length ℓ should be

$$g(\ell) \stackrel{?}{=} \sum_{n=1}^{\ell} \frac{1}{n!} \sum_{\ell_1+\ell_2+\ell_n=\ell} D(\ell_1) \ldots D(\ell_n)$$

Contributions to $g(12)$:



Problem! Overcounting!

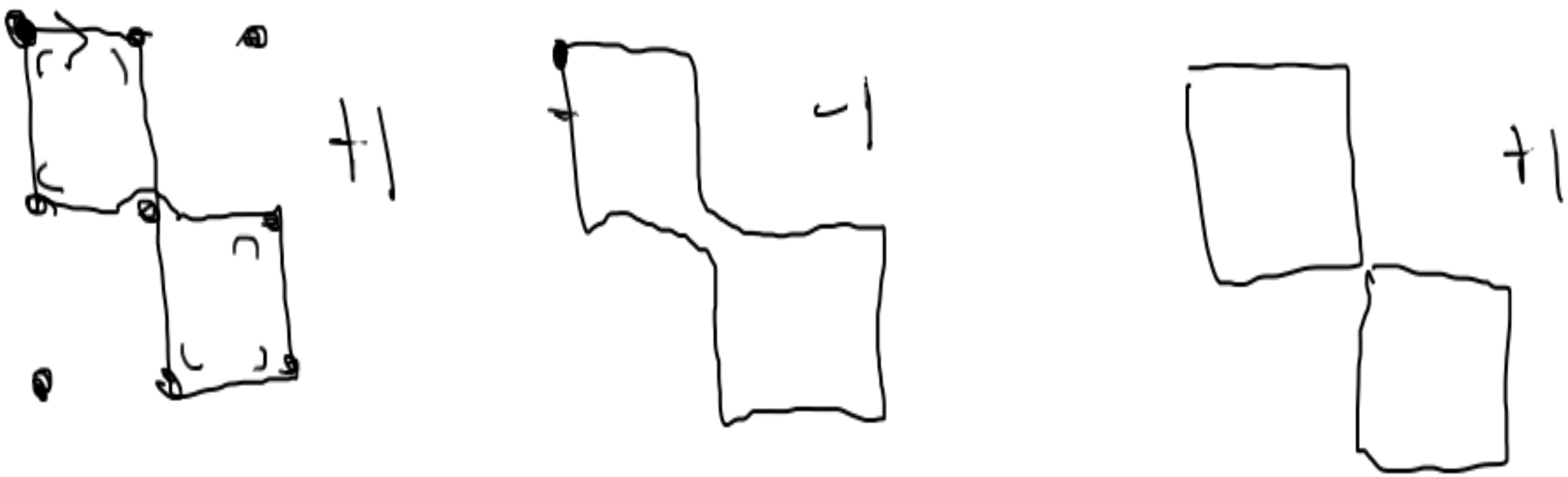


To solve this: add a handy complication:

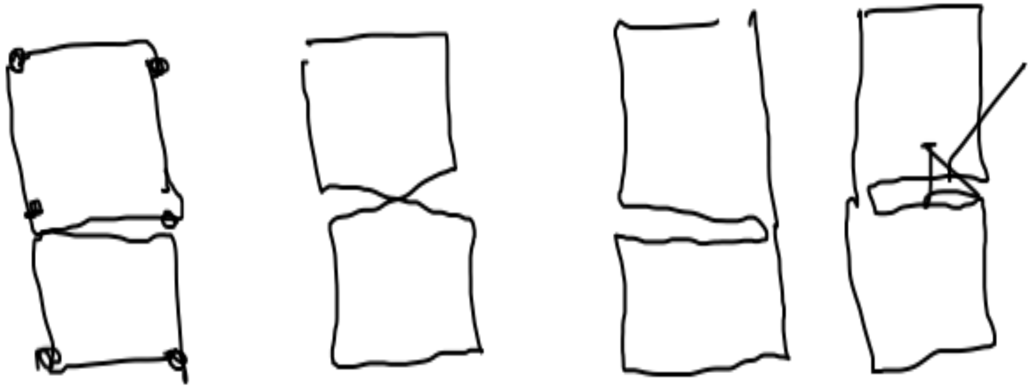
when particle turns left, factor $e^{i\pi/4}$

" " " right, " $e^{-i\pi/4}$

Phase factor of path = $\prod$ of phase factors along path



Other pattern with problem:



! can't go twice over a link

Factors: +1, +1, −1, −1

To sum over paths: let a matrix do the work.

Define M with elements M_{ij} nonzero if sites i & j are linked by a single segment.

So if we put $M_{ij} = 1$ if i & j are nearest-neigh.,

then $[M^l]_{ij}$ = # of paths of length l joining i & j. [with overcounting]

Complication from overcounting:

actually need to implement phases.

Let $m^{\alpha\beta}_{\tilde{i},\tilde{j}}$ with $\alpha,\beta = 0,1,2,3$

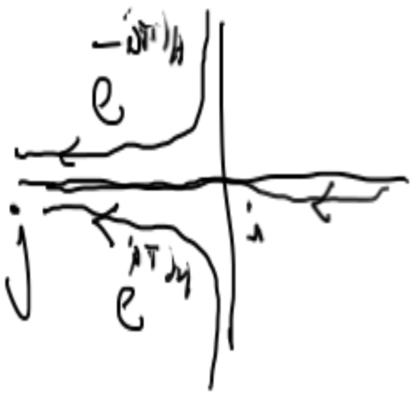
be (for fixed i, j nearest neighbours) a 4×4 matrix

α labels entry direction

$m^{\alpha\beta}_{\tilde{i}\tilde{j}}$ — β labels the exit dirn; use $\tilde{i}-\tilde{j}$ as label

Convention: 0 = east, 1 = N, 2 = West, 3 = South

Ex.: $\tilde{m}_{(+1,0)}$



$$m_{(1,0)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i\pi/4} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & e^{-i\pi/4} & 0 \end{pmatrix}$$

We now have a big matrix:

$$M = \begin{array}{cc} & \xrightarrow{4\times\#\text{sites}} \\ {\scriptstyle 4\times\#\text{sites}}\downarrow & \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \end{array} = \begin{array}{cc} & \xrightarrow{\#\text{sites}} \\ {\scriptstyle \#\text{sites}}\downarrow & \begin{pmatrix} & & \\ & m_{ij} \leftarrow 4\times 4 & \\ & & \end{pmatrix} \end{array}$$

Now: $(M^l)^{\alpha\alpha}_{ii}$ = #paths starting at i, leaving in dir^n α, & coming back after l steps.

We thus have $D(l) \equiv -\frac{1}{2l} \mathrm{Tr}\, M^l$

& $$Z = [\cosh \beta \varepsilon]^{\frac{NZ}{2}} 2^N \left[1 + \sum_{l=1}^{\infty} v^l \sum_{n=1}^{l} \frac{1}{n!} \sum_{l_1+\dots+l_n=l} D(l_1)\dots D(l_n) \right]$$

$$= [\;]^{\frac{NZ}{2}} 2^N \Big[\underbrace{1 + \sum_{n=1}^{\infty} \frac{1}{n!} \Big(\sum_{l=1}^{\infty} D(l) v^l \Big)}_{\exp \sum_{l=1}^{\infty} D(l) v^l} \Big]$$

Rest: eigenvalues of M (really: m), ...