# From Path Integrals to Field Theory 2018
# Take-home exam

To be handed back in on Friday 8 June 2018 (either by sliding your sheets under my office door C4.262, or by email to J.S.Caux@uva.nl or prof@jscaux.org).

- Please **be explicit** in your answers. I cannot give you points for things I can't/don't see! You are welcome to deliver your answers in pdf'ed LaTeX. If you scan handwritten notes, please ensure clarity and a reasonable size for your digital scan (don't flood my inboxes).
- FPITFT class notes are **allowed**. You are **not allowed** to use other sources.
- You are **NOT** allowed to collaborate with anybody. I expect originality from everybody.
- This exam consists of two problems, both of which you should do.

## Useful formula

Besides the formulas from the notes, you might also find that the **error function**

$$\text{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x dx \; e^{-x^2}$$

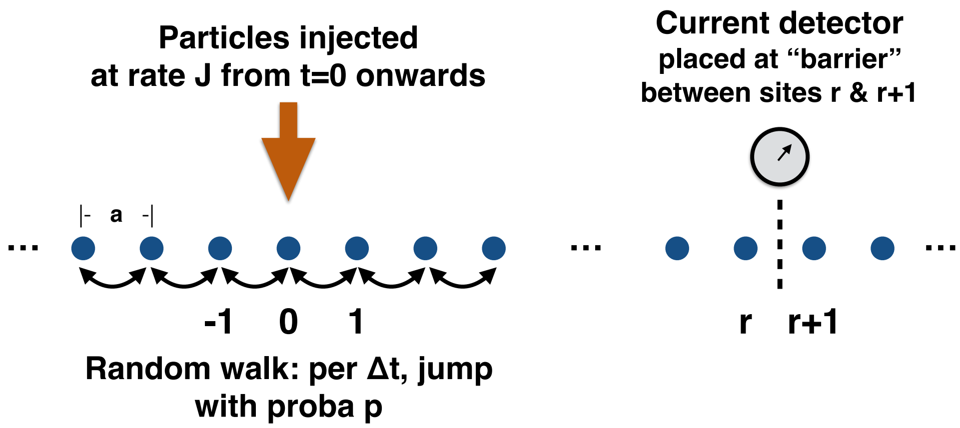
and its sibling the **complementary error function**

$$\text{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty dx \; e^{-x^2} = 1 - \text{erf}(x)$$

might come in handy, either somewhere here, or if not then elsewhere during your hopefully long, happy and productive life.

## 1. Diffusive current in one dimension (60 pts)

You have at your disposal a one-dimensional crystal lattice with intersite spacing $a$, extending infinitely in both directions (so you label the sites with integers). Since you're really an experimentalist at heart, you decide to perform some simple transport experiment using this special material. At $t = 0$, you start injecting particles at site 0 at a rate of $J$ particles per unit of time. You place a current detector between sites $r$ and $r + 1$ so it counts the number of particles jumping from $r$ to $r + 1$, minus the number of particles jumping from $r + 1$ to $r$, per unit of time. This current is obviously time dependent; we shall write it as $J_{r,t}$. See the Figure for a general illustration of the setup.




The particles you inject are chosen because they have the following properties:

- they do not interact with each other (you can have as many of them as you want on a site);
- they perform a random walk defined by the following rules: at each interval $\delta t$, they have a probability $p$ (with $0 \leq p \leq 1$) of hopping (they therefore stay in place with probability $1 - p$). If they hop, they go left or right with equal probability.
- your current detector can see them without affecting their dynamics (namely: your particles' random walk is unaffected by the detector).

**a) (20 pts)** Start by considering an individual particle's random walk on the lattice. Let $P_{r,t|r',t'}$ be the probability of finding it at site $r$ at time $t$, given that it was at site $r'$ at time $t'$. Write down the one-step time evolution equation for this probability. Solve this equation using an appropriate Fourier series.

**b) (10 pts)** Write an expression for the current $J_{r,t}$, in terms of probabilities $P_{r',t'|r'',t''}$ (using any sum/product/ratio you need of these functions evaluated at any space/time point you wish).

**c) (30 pts)** Take a scaling limit of your results of a), and of the equation for the current you obtained in b). Calculate the current $J(r,t)$ in the scaling limit, for all values of position and time. Give a plot of the result, specifying the appropriate space and time scales over which it is nontrivial.

## 2. Flea dynamics (40 pts)

Fleas (siphonaptera), despite their parasitic nature, are amazing insects. Besides simply walking around, they can also jump distances up to many dozen times their body size. Your government, in a bid to save the world from future plagues, has tasked you with the secret mission of understanding how fleas can propagate by hopping, so their evil plans for mankind can be defeated.

To start working towards a realistic model of flea dynamics, what you first want to quantify is the effect of being able to walk/jump various distances at each time step. Let us thus define a variation of our basic random walk problem. For ease of treatment, we consider a one-dimensional lattice with lattice spacing $a$ on which our flea is taken to live.

- **rule 1**: at each time interval $\delta t$, the flea walks one lattice step with probability $p_1$, **or** jumps two steps in one lattice direction with probability $p_2$. Of course, $0 \le p_i \le 1$ for $i = 1, 2$, with $0 \le p_1 + p_2 \le 1$.
- **rule 2**: the direction of a step or of a jump is uniformly distributed among all possible directions (here: 2, left or right).

**a) (10 pts)** Solve this problem exactly, by giving the exact probability distribution $P_{r_1,t_1|r_0,t_0}$ of finding the flea at $r_1$ at time $t_1$, given that it was at $r_0$ at $t_0$.

**b) (10 pts)** Take a scaling limit of your result, and give the analytic form of the probability density distribution $p(r_1, t_1|r_0, t_0) = \lim a^{-1} P_{r_1,t_1|r_0,t_0}$.

**c) (10 pts)** In the scaling limit, for a total interval of time $t$, we could define a seemingly equivalent problem by saying that the flea walks a step at a time for a time interval $p_1 t$, after which it jumps two steps at a time for a time interval $p_2 t$ (followed or preceded, as you wish, by staying still for time $(1 - p_1 - p_2)t$). Composing the appropriate result for each of these individual random walks (c.f. notes), does this lead you to the same result as in b)?

**d) (10 pts)** Consider now the more general problem of a flea able to jump up to a distance $J$. To a jump of length $j$ (with $j = 1, ..., J$), you associate probability $p_j$ with $0 \le \sum_{j=1}^{J} p_j \le 1$ (for convenience, you can define $j = 0$ as staying in place, with associated probability $p_0$, so $1 - p_0 = \sum_{j=1}^{J} p_j$). Give the exact solution of this problem on the lattice, as well as its scaling limit form.

**Bonus question:** Show that the scaling limit form you obtained in d) equals what you would obtain for the seemingly equivalent problem of the flea spending $p_1 t$ jumping distance one, followed by time $p_2 t$ jumping distance two, ... followed by time $p_J t$ jumping distance $J$.