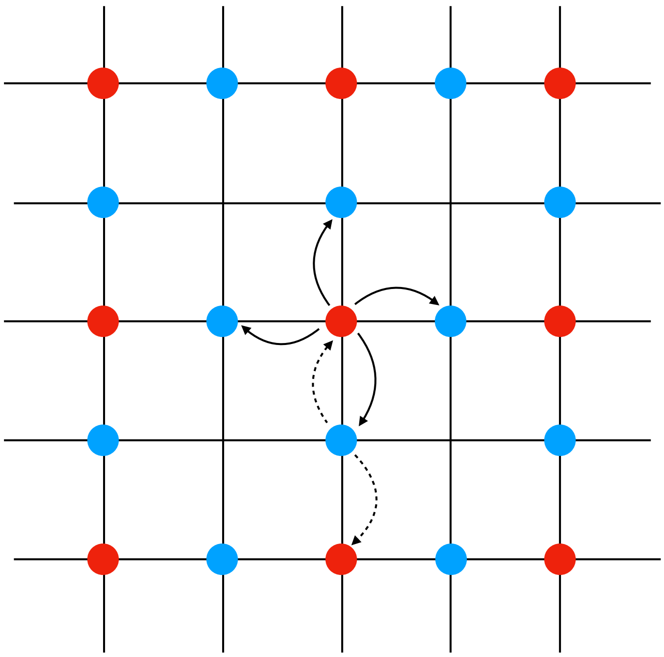
From Path Integrals to Field Theory 2019
Take-home exam

To be handed back in on Monday 3 June 2019 (either by sliding your sheets under my office door C4.262, or by email to J.S.Caux@uva.nl or prof@jscaux.org).

- Please **be explicit** in your answers. I cannot give you points for things I can't/don't see! You are welcome to deliver your answers in pdf'ed LaTeX. If you scan handwritten notes, please ensure clarity and a reasonable size for your digital scan (don't flood my inboxes).
- FPITFT class notes are **allowed**. You are **not allowed** to use other sources.
- You are **NOT** allowed to collaborate with anybody. I expect originality from everybody.
- This exam consists of three problems, all of which you should do.
- Sub-questions marked with * are particularly challenging and don't give many points.

1. Diffusion on the Lieb lattice (40 pts)

The **Lieb Lattice** is a simple but remarkably interesting two-dimensional structure, named after the famous theorist Elliott Lieb. It consists of a square lattice from which half the atoms of one sublattice have been removed, leaving a so-called edge-centered square lattice (see the illustration).



In this problem, we consider a random walk on this lattice. The rules of this process are simply formulated as follows:

- **Initial condition**: at time $t = 0$, our particle is on a red site taken to be the origin $\boldsymbol{r} = 0$.
- During time intervals $t = 2n\delta t$ to $t = (2n+1)\delta t$ (with n integer), the particle takes one step from red to blue, with uniform probability among the 4 possibilities (full arrows in the figure).
- During time intervals $t = (2n+1)\delta t$ to $2(n+1)\delta t$, the particle takes one step from blue to red, with uniform probability among the 2 possibilities (dashed arrows in the figure).

a) (10 pts) Write down the discrete time evolution equation for this problem. For this purpose, you can specialize to discrete time evolution from "even" times $t = 2n\delta t$ to "even" times only. Is this process a Markov process?

b) (20 pts) Using a judiciously-chosen Fourier transform, write down the exact solution for the probability distribution function $P_{\mathbf{r},t|0,0}$ of finding the particle at $\mathbf{r}$ at time t, given that it was at the origin at $t = 0$.

c) (10 pts) Define an appropriate continuum limit, and rewrite your distribution in this limit. Explicitly give the value of the diffusion constant.

2. Lévy flight (45 pts)

In all diffusion processes we have seen, we have ended up with a scaling relation of the form (distance) $\sim$ (time)$^{1/2}$, which is characteristic of normal diffusion. A natural question to ask is: are other scalings possible?

We will consider a special random walk in which hopping on many length scales is allowed. This is meant to model many "real-life" circumstances. For example: in their search for prey, sharks move around locally in a random walk; if they don't find anything to eat, they sometimes "migrate away" to another place (as if they were taking a random step, but with much longer step size), and start a new short-scale random walk.

For simplicity, we will consider a one-dimensional lattice of uniform spacing a. We will consider a random walk with one-time-step equation

$$P_{r,t+\delta t} = \sum_{l=-\infty}^{\infty} h_l P_{r-la,t}$$

in which h_l represents the probability of jumping a distance la.

We will use a scaling form for the step size probabilities. Consider a jump factor j (which you can take to be an integer > 1). We will allow jumps of "size" $j, j^2, j^3, \ldots$. Defining a probability scale s (with $0 < s < 1$), we will set that a jump which goes j times as far, occur s times less often. In other words, we will set $h_{l=j^n} \propto s^n$.

a) (10 pts) Writing

$$h_l = N(s) \sum_{n=0}^{\infty} s^n \left[\delta_{l,-j^n} + \delta_{l,j^n}\right]$$

to represent the jumping probabilities, show that the normalization constant must be $N(s) = \frac{1-s}{2}$.

b) (10 pts) We have not yet specified any values for the jump factor j or the probability scale s. Before we do that, we should ask ourselves what is the "root mean square" deviation in the hopping distance. Said otherwise, calculate

$$\sigma = \sqrt{\langle l^2 \rangle - \langle l \rangle^2},$$

as a function of j and s, where the moments are defined as $\langle l^m \rangle \equiv \sum_{l=-\infty}^{\infty} h_l l^m$.

What is the fundamental difference between the cases $sj^2 < 1$ and $sj^2 > 1$? Do you expect to see normal diffusion if $sj^2 < 1$?

c) (10 pts) Getting back to the random walk equations: introducing a Fourier representation, show that the one-time-step equation can be solved for all times as

$$P_{k,t+n\delta t} = \left[f_{j,s}(k)\right]^n P_{k,t}$$

where you should obtain the explicit form $f_{j,s}(k) = (1-s) \sum_{n=0}^{\infty} s^n \cos[j^n ak]$.

For your edification: this is called the Weierstrass function, and it has the remarkable characteristic of being *everywhere continuous but nowhere differentiable* (we will **not** make use of these fascinating facts here).

d) (5 pts) Show that the function $f_{j,s}$ obtained above obeys the remarkable scaling law

$$f_{j,s}(jk) = \frac{1}{s} f_{j,s}(k) - \frac{1}{s}(1-s)\cos ak$$

e*) (5 pts) Since we will want to go towards some scaling limit, we will be interested in the low-momentum limit of our functions. Let us now focus on the most interesting region defined by $sj^2 > 1$, in which hopping occurs on all scales up to infinity. For this range of parameters, the function $f_{j,s}(k)$ turns out to have a very complicated momentum dependence at small momenta: its behaviour is not the conventional integer-power series!

Show rather that the following fractional-power scaling form is consistent with the scaling law in d):

$$f_{j,s}(k) = \lambda_0 + c|k|^\mu + O(k^2)$$

where you should give the values for λ_0 and μ (you don't have to give c).

f*) (5 pts) Try to define a consistent scaling limit, and to show that the time evolution of the probability distribution function is given by an equation of the form (you don't have to compute the constant $\tilde{D}$)

$$\partial_t P_k = \tilde{D}|k|^\mu P_k$$

where μ is the fractional power obtained above. What does that mean for the scaling relation between space and time for this type of diffusion process?

Such scale-free random walks represent a fundamentally different universality class than the normal diffusion ones. They are known as **Lévy flights**.

3. Feynman propagator (15 pts)

A free particle living in one dimension is initialized in a state with wavefunction

$$\Psi(q, t = 0) = \frac{e^{ip_0 q - (q-q_0)^2/2a}}{\sqrt{\pi a}}$$

Using the free particle propagator we have seen in class, compute the wavefunction at time $t_f > 0$. Draw a sketch of the wavefunction amplitude. Where is the most likely place to find the particle at time t_f? (said otherwise: where is the peak in the amplitude situated?)